Leonel Zuaznábar Moliner

A VERSION WITH CROSSING OF THE RANDOM DIRECTED FOREST AND ITS CONVERGENCE TO THE BROWNIAN WEB

A thesis presented to the Federal University of Rio de Janeiro in fulfillment of the thesis requirement for the degree of Doctor in Statistic.

Advisor: Glauco Valle

Rio de Janeiro - Brazil
2017

Leonel Zuaznábar Moliner

A VERSION WITH CROSSING OF THE RANDOM DIRECTED FOREST AND ITS CONVERGENCE TO THE BROWNIAN WEB

Rio de Janeiro, __________ __________ of *2017*.

____

Prof. **Glauco Valle**, Ph.D., UFRJ - Brazil.

____

Prof. **Luiz Renato Fontes**, Ph.D., USP - Brazil.

____

Prof. **Maria Eulalia Vares.**, Ph.D., UFRJ - Brazil.

____

Prof. **Leandro Pimentel** , Ph.D., UFRJ - Brazil.

Leonel Zuaznábar Moliner

A VERSION WITH CROSSING OF THE RANDOM DIRECTED FOREST AND ITS CONVERGENCE TO THE BROWNIAN WEB

Prof. **Cristian Favio Coletti**, Ph.D., UFABC - Brazil.

To my parents, Taty and Luis.

Acknowledgements

I would like to thank to: my advisor Glauco Valle, and the professors Maria Eulalia Vares and Lenadro Pimentel, by all the math they taught me in the last four years; professor Luiz Renato Fontes, by help me in this project with very useful coments and suggestion; family and friends for the support and the love given to me.

Abstract

The Brownian web is the collection of graphs of coalescing one-dimensional Brownian motions, starting from all possible starting points in one plus one-dimensional space/time. Several authors have studied convergence in distribution to the Brownian web, under diffusive scaling of Markovian random walks. In a paper by R. Roy, K. Saha and A. Sarkar, convergence to the Brownian web is proved for a system of coalescing random paths which are not Markovian: the Random Directed Forest (RDF). In view of the fact that paths in the RDF do not cross each other before coalescence, we study here a version where crossing could happen and prove convergence to the Brownian web. This show how broad is the class of coalescing paths that converges to the Brownian web; and provides an example of how the techniques to prove convergence to the Brownian web for systems allowing crossings, can be applied to non-Markovian systems.

TABLE OF CONTENTS

CHAPTER
ONE

INTRODUCTION

The study of the Brownian web started in 1979 with Arratia's doutoral thesis ([1]). He constructed a process of coalescing one-dimensional Brownian motions starting from every point in $\mathbb{R}$ at time zero. Later Arratia, motivated by asymptotic of one-dimensional voter models, try to generalize this construction in [2] - an unpublished manuscript - to a process of coalescing Brownian motions, starting from every point in space and time $\mathbb{R} \times \mathbb{R}$. The Brownian web was not studied again until Tóth and Werner in [15] - motivated by the construction of the continuum self-repelling motion - constructed a version of the Brownian web. Years later Fontes, Isopi, Newman and Ravishankar gave a different construction in [8], [10], and used the first time the term Brownian web. Also in these works the authors proved a criterium of convergence in distribution to this object and used it to obtain the scaling limit of a system of coalescing random walks.

After [8] and [10] several authors have studied convergence in distribution to the Brownian web for different processes, for instance [4], [6], [7], [10], [11], [13] to mention some works. The aim most of these papers is the understanding of the universality class associated to the Brownian web. This was a breakthrough because it was an important question in probability theory about how to characterize properly the convergence of systems of coalescing random walks which started to be studied by Arratia. From [8] and [10] the question about the universality class for the Brownian web arises as important one, since many important systems of coalescing random paths related to applications of probability theory are more complicated; for instance they may have long range dependence, they are not necessarily independent before coalescence and they are not necessarily Markovian. See for instance the Poisson Tree in [7], the Drainage Network Model studied in [5] and [6], the Random Directed Forest studied in [13] or the Direct Spaning Forest in [3], where the authors made a conjecture about the convergence to a transformation of the Brownian web in its Remark 4.9 that was proved for a similar system in [11].

The main result of this thesis is Theorem 1.2.1. Something that we do not have in previous works and show that the class of coalescing paths that converges to the Brownian web is very wide.

In the remaining sections of this chapter we will present, at first, the space where Fontes, Isopi, Newman and Ravishankar defined the Brownian web. In the same Section 1.1 we enunciate the theorem that characterizes the Brownian web proved in [10] and a convergence criterium. In Section 1.2 we will introduce the Random Direct Forest studied by Roy, Saha and Sarkar in [13], and define a version where paths could cross each other. At the end of this section we enunciate the main result of the thesis.

1.1 The Brownian web

In this section we will recall the space used in [8] and [10] to define the Brownian web and we will present the theorem proved there to characterize it. Also we will enunciate a variation of the convergence criterium proved in these papers.

Roughly speaking - as Fontes, Isopi, Newman and Ravishankar said in [10] - the Brownian web is the collection of graphs of coalescing one-dimensional Brownian motions starting from all possible starting points in one plus one-dimensional space/time. Even though it is difficult to think in coalescing Brownian motions starting at any space/time point of $\mathbb{R}^2$, fortunately - as the following Theorem 1.1.1 says - the Brownian web is fully determined by a countable number of coalescing Brownian motions starting from a countable dense subset D of $\mathbb{R}^2$.

In [10] the authors considered $(\bar{\mathbb{R}}^2, \rho)$, a completion of $\mathbb{R}^2$ under the metric ρ defined as

$$\rho\big((x_1,t_1),(x_2,t_2)\big) := \Big|\frac{\tanh(x_1)}{1+|t_1|} - \frac{\tanh(x_2)}{1+|t_2|}\Big| \vee \big|\tanh(t_1) - \tanh(t_2)\big|.$$

We may think $\bar{\mathbb{R}}^2$ as the image of $[-\infty,\infty]\times[-\infty,\infty]$ under the mapping

$$(x,t) \to \big(\Phi(x,t),\Psi(t)\big) := \Big(\frac{\tanh(x)}{1+|t|}, \tanh(t)\Big). \tag{1.1.1}$$

For $t_0 \in [-\infty,\infty]$, let $C[t_0]$ be the set of functions from $[t_0,\infty]$ to $[-\infty,\infty]$ such that $\Phi\big(f(t),t\big)$ is continuous. Then define

$$\Pi = \bigcup_{t_0\in[-\infty,\infty]} C[t_0]\times\{t_0\}.$$

For (f,t_0) in Π, let us denote $\widetilde{f}$ the function that extends f to all $[-\infty,\infty]$ by setting it equal to $f(t_0)$ for $t \leq_0$. Take

$$d\big((f_1,t_1),(f_2,t_2)\big) = \Big(\sup_{t\geq t_1\wedge t_2} |\Phi(\widetilde{f_1}(t),t) - \Phi(\widetilde{f_2}(t),t)|\Big) \vee |\Psi(t_1)-\Psi(t_2)|.$$

Let now $\mathcal{H}$ denote the set of compact subset of (Π,d) with the Hausdorff metric $d_{\mathcal{H}}$,

$$d_{\mathcal{H}}(K_1,K_2) := \sup_{g_1\in K_1}\inf_{g_2\in K_2} d(g_1,g_2) \vee \sup_{g_2\in K_2}\inf_{g_1\in K_1} d(g_1,g_2),$$

for K_1, K_2 non-empty sets in $\mathcal{H}$. $\mathcal{F}_{\mathcal{H}}$ is the Borel σ-field induced by $(\mathcal{H}, d_{\mathcal{H}})$.

The existence of a $(\mathcal{H},\mathcal{F}_{\mathcal{H}})$- valued random variable, called by Fontes, Isopi, Newman and Ravishankar as the Brownian web, was proved in the Theorem 2.1 in [10]. We will present it in the following theorem.

Theorem 1.1.1. *There exists a $(\mathcal{H},\mathcal{F}_{\mathcal{H}})-$ valued random variable $\mathcal{W}$ whose distribution is uniquely determined by the following three properties:*

(i) For any deterministic point (x,t) in $\mathbb{R}^2$ there exists almost surely an unique path $\mathcal{W}_{x,t}$ starting from (x,t).

(ii) For any deterministic $n, (x_1,t_1),\ldots,(x_n,t_n)$ the joint distribution of $\mathcal{W}_{(x_1,t_1)}, \ldots, \mathcal{W}_{(x_n,t_n)}$ is that of coalescing Brownian motions.

(iii) For any deterministic, dense countable subset $\mathcal{D}$ of $\mathbb{R}^2$, almost surely, $\mathcal{W}$ is the closure in $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$ of $\{\mathcal{W}_{x,t} : (x,t) \in \mathcal{D}\}$.

The next Theorem 1.1.2 is a criterium of convergence to the Brownian web. This theorem is a variation of the Theorem 2.2 proved in [10] which can be found as the Theorem 1.4 in [12]. These theorems (Theorem 2.2 in [10] and Theorem 1.4 in [12]) have been the principal tools to prove the convergence to the Brownian web for many different kind of coalescing system.

Theorem 1.1.2. *Let $\{\mathcal{Y}_n\}_{n\geq 1}$ be a sequence of $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$-valued r.v. We have that $\{\mathcal{Y}_n\}_{n\geq 1}$ converges to the Brownian web if the following conditions are satisfied:*

(I) There exist some deterministic countable dense subset of $\mathbb{R}^2$, let us called D, and $\theta_n^y \in \mathcal{Y}_n$ for any $y \in D$ satisfying: for any deterministic $y_1, \ldots, y_m \in D, \theta_n^{y_1}, \ldots, \theta_n^{y_m}$ converge in distribution as $n \to \infty$ to coalescing Brownian motions starting in $y_1, \ldots, y_m$.

(B) $\forall \beta > 0, \limsup_n \sup_{t>\beta} \sup_{t_0, a \in \mathbb{R}} \mathbb{P}\big[|\eta_{\mathcal{Y}_n}(t_0, t, a-\epsilon, a+\epsilon)| > 1\big] \to 0$ as $\epsilon \to 0^+$, where

$$\eta_{\mathcal{Y}_n}(t_0, t, a, b) = \big\{y \in \mathbb{R} \times \{t_0 + t\} : \text{are touched by paths in } \mathcal{Y}_n \text{ which also touch some point in } [a,b] \times \{t_0\}\big\}.$$

(E) For some $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$-valued r.v. Y and $t > 0$ take Y^{t^-} as the subset of paths in Y which start before or at time t. If Z_{t_0} is the subsequential limit of $\{\mathcal{Y}_n^{t_0^-}\}_{n\geq 1}$ for any t_0 in $\mathbb{R}$, then for all t, a, b in $\mathbb{R}$ with $t > 0$ and $a < b$ we get

$$\mathbb{E}\big[|\eta_{Z_{t_0}}(t_0, t, a, b)|\big] \leq 1 + \frac{b-a}{\sqrt{\pi t}}.$$

(T) Let $\Lambda_{L,T} := [-L, L] \times [-T, T] \subset \mathbb{R}^2$ and for $(x_0, t_0) \in \mathbb{R}^2$ and $\rho, t > 0$,

$$R(x_0, t_0; \rho, t) := [x_0 - \rho, x_0 + \rho] \times [t_0, t_0 + t].$$

For $K \in \mathcal{H}$ define $A_{\mathcal{K}}(x_0, t_0; \rho, t)$ to be the event that K contains a path touching both $R(x_0, t_0; \rho, t)$ and the right or the left boundary of the rectangle $R(x_0, t_0; 20\rho, 4t)$. Then for every $\rho, L, T \in (0, \infty)$

$$\frac{1}{t} \limsup_{n\to\infty} \sup_{(x_0,t_0)\in\Lambda_{L,T}} \mathbb{P}\Big[A_{\mathcal{Y}_n}(x_0, t_0; \rho, t)\Big] \to 0 \text{ as } t \to 0^+.$$

1.2 A version of the Random Directed Forest and the main result

R. Roy, K. Saha and A. Sarkar in [13] study the Random Directed Forest which is a system of coalescing space and time random paths on $\mathbb{Z}^2$ as we now describe. Suppose that the first coordinate of a point in $\mathbb{Z}^2$ represents space and the second one time. We start a space/time random path in each point of $\mathbb{Z}^2$. The path starting at u in $\mathbb{Z}^2$ evolves as follows: every point in $\mathbb{Z}^2$ is open with some probability p or closed with $1-p$ independently of each other. We say that a point $v = (\tilde{x}, \tilde{t}) \in \mathbb{Z}^2$ is above $u = (x, t)$ if

$\tilde{t} > t$. If the path is at space/time position (v, t) then it jumps to the nearest open point in the L_1 norm above (v, t) if this nearest open point is unique. If it is not unique then the a choice is made uniformly to decide where the path has to jump to (see the Figure 1.1). Note that two paths cannot cross each other and after one step it is possible to know something about the future, that is to say, maybe we know if some points above the current position of the path are open or closed. That is why we get a system of non-Markovian random paths.

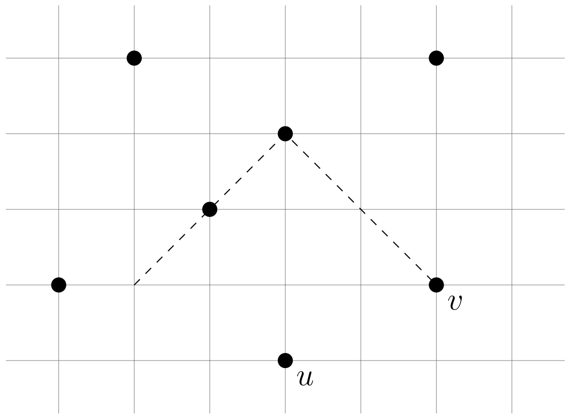

Figure 1.1: Open points in $\mathbb{Z}^2$ are marked by black dots. Note that the closest open points above u in the L_1 distance are those connected by the dashed line. Hence the path starting at u moves to one of these points connected by the dashed line chosen uniformly among them; for instance it could be v.

In [13] the authors proved that under diffusive scaling, the closure of linearly interpolation trajectory induced by each discrete random path, in some space where the Brownian web is defined, converges in distribution to the Brownian web. Our initial aim was to consider a generalization of the random directed forest that allows crossings before coalescence analogous to the generalized drainage models studied in [6]. This could be made if we do not impose the necessity that the jump should be made to the nearest open above position. Although we get a well defined system, we were not able to prove convergence to the Brownian web in this case. The problem here is to build a regeneration structure similar to that presented in [13], which is the way around the non-Markovianity of paths in GRDF.

We will define a model which is slightly different from the Random Directed Forest and consider a generalization of it that allows crossing before coalescence. Suppose now that in each point u in $\mathbb{Z}^2$ we have a random variable W_u such that $\{W_u; u \in \mathbb{Z}^2\}$ is an i.i.d. family of random variables in the set of positive integers. We will call the k-th level of some $u = (u(1), u(2))$ in $\mathbb{Z}^2$ the following set

$$L(u, k) := \{v = (v(1), v(2)) \in \mathbb{Z}^2 : v(2) > u(2) \text{ and } ||v - u||_1 = k\}, \tag{1.2.1}$$

where for any $u \in \mathbb{Z}^2$, $||u||_1 := |u(1)| + |u(2)|$. A level $L(u, k)$ is called open if it has at least one open point. Now consider that the path not necessarily move to the nearest open point above him, but to the the highest open point in the W_u-th open level. If the path has two options to jump to, it makes an uniform choice to decide. See the Figure 1.2 for an example.

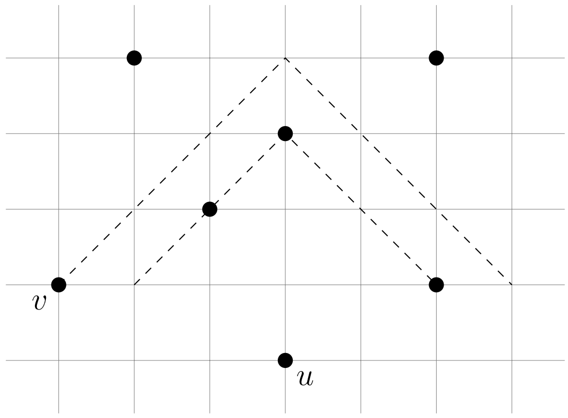

Figure 1.2: Note that points connected by the dashed lines are the first and the second open levels of u. If $W_u = 2$ for instance, the path starting in u jumps to v.

As the Directed Random Forest we now have a system of non-Markovian walks but in this case the paths can cross each other. Our goal is to prove convergence in distribution to the Brownian web under diffusive scaling for the closure of the system of linearly interpolated paths, see Theorem 1.2.1.

Let us define formally the process. First let us fix some notation that will be used in the text. Consider the following random variables:

(i) Let $(W_u)_{u\in\mathbb{Z}^2}$ be a family of i.i.d. random variables with finite support on $\mathbb{N}$ such that $\mathbb{P}[W_u = 1] > 0$. From now on we will use the latter K as a constant such that $\mathbb{P}[W_u \leq K] = 1$. Denote by $\mathbb{P}_W$ the induced probability on $\mathbb{N}^{\mathbb{Z}^2}$.

(ii) Let $\{U_v; v \in \mathbb{Z}^2\}$ be a family of i.i.d. Uniform random variables in $(0, 1)$. Denote by $\mathbb{P}_U$ the induced probability on $(0, 1)^{\mathbb{Z}^2}$.

We suppose that the two families above are independent of each other and thus have a joint distribution given by the product probability $\mathbb{P} := \mathbb{P}_W \times \mathbb{P}_U$ on the space $\mathbb{N}^{\mathbb{Z}^2} \times (0, 1)^{\mathbb{Z}^2}$.

Now fix some $p \in (0, 1)$. We write $u = (u(1), u(2))$ for $u \in \mathbb{Z}^2$ and call open the points in $V := \{u \in \mathbb{Z}^2; U_u < p\}$ and closed the points in $\mathbb{Z}^2 \setminus V$. We will denote the index of the r-th open level of u - recall the definition of a level in (1.2.1) - by $h(u, r)$, i.e.

$$h(u, r) := \inf\Big\{k \geq 1 : \sum_{j=1}^{k} \mathbb{1}_{\{L(u,j)\cap V \neq \emptyset\}} = r\Big\}.$$

For $u \in \mathbb{Z}^2$ denote by $X(u)$ the unique (almost surely) point in $L(u, h(u, W_u)) \cap V$ such that for every $w \in L(u, h(u, W_u)) \cap V$ either $X(u)$ is above w or $U_{X(u)} > U_w$. Let us define the sequence $\{X_n(u)\}_{n\geq 0}$ as,

$$X_0(u) := u \text{ and } X_n(u) := X(X_{n-1}(u)) \text{ for } n \geq 1.$$

Now define $\pi^u : [v(2), \infty] \to [-\infty, \infty]$ as $\pi^u(X_n(u)(2)) := X_n(u)(1)$, linearly interpolated in $[X_n(u)(2), X_{n+1}(u)(2)]$ and $\pi^u(\infty) = \infty$. Let us denote the set of paths by

$$\mathcal{X} := \big\{(\pi^v, v(2)); v \in \mathbb{Z}^2\big\}. \tag{1.2.2}$$

The system $\mathcal{X}$ is the modified Random Directed Forest which is the main object of study in this paper. From now on we call it the Generalized Random Directed Forest (GRDF).

We are interested in the diffusive rescaled GRDF. So let $\gamma > 0$ and $\sigma > 0$ be some fixed normalizing constants to be determined latter, $u \in \mathbb{Z}^2$ and $n \in \mathbb{N}$. Let us define

$$\pi_n^u(t) := \frac{\pi^u(n^2\gamma t)}{n\sigma} \text{ for } t \in [0, \infty),\ \pi_n^u(\infty) = \infty$$

and

$$\mathcal{X}_n := \left\{(\pi_n^v, v(2)) : v \in \mathbb{Z}^2\right\}. \tag{1.2.3}$$

The system of coalescing paths $\mathcal{X}_n$ is the rescaled GRDF and our aim is to prove that its closure converges to the Brownian web as n goes to infinity.

Theorem 1.2.1. *There exist positive constants γ and σ such that $\bar{\mathcal{X}}_n$, the closure of $\mathcal{X}_n$ in $(\mathcal{H}, d_{\mathcal{H}})$, converges in distribution to the Brownian web as n goes to infinity.*

CHAPTER

TWO

WELL POSEDNESS, RENEWAL TIMES AND COALESCING TIME

In this chapter we first prove that $\bar{\mathcal{X}}_n$, the closure of $\mathcal{X}_n$, is almost surely a compact set in $(\mathcal{H}, d_{\mathcal{H}})$ for all $n \geq 1$. Therefore we are indeed working with random elements of $(\mathcal{H}, d_{\mathcal{H}})$ where the Brownian web is defined.

In Section 2.2 we prove the existence of regeneration times where the random paths in the GRDF have no information about the future. The idea of using regeneration times came from [13] and is fundamental since the paths seen at these times have the Markov property. However we are not able to get the existence as they did it, since in our case the paths get into regions which have been observed before, something that the Random Directed Forest model does not allow do and it is used in the proof given in [13]. So we will follow a different approach here.

In Section 2.3 we will obtain an upper bound on the tail probability of the coalescence time of two paths in $\mathcal{X}$. This is a central estimate related to convergence to the Brownian web. Related to other processes see [5],[6] and [13] for instance. The main ideas used here to get the bound come from these three works, although it is not a straightforward application of the techniques used before. Here we have another important difference with the Random Directed Forest studied in [13] because of the possibility of the paths to cross each other before coalescence. This property does not allow us to follow the proof in [5] as done in [13]. We will need the ideas used in [6], where the authors work with a system that allows crossing, to control the coalescence time.

2.1 Well posedness

Lemma 2.1.1. *Let N be some positive integer random variable and $(\zeta_n)_{n\geq 1}$ a non-negative sequence of identically distributed random variables. If for some $k \geq 1, \delta > 0$ and $l > \frac{(k+2)(1+\delta)}{\delta}$ we have $\mathbb{E}[\zeta_1^{k(1+\delta)}]$ and $\mathbb{E}[N^l]$ finite, then for $S := \sum_{n=1}^{N} \zeta_n$ we get that $\mathbb{E}[S^k]$ is also finite.*

Proof. We have that $0 \leq S \leq N \max_{1\leq j\leq N} \zeta_j$ what implies that

$$S^k \leq N^k \max_{1\leq j\leq N} \zeta_j^k \leq N^k \sum_{j=1}^{N} \zeta_j^k.$$

Hence

$$\mathbb{E}\big[S^k\big] \leq \mathbb{E}\Big[N^k \sum_{j=1}^{N} \zeta_j^k\Big] = \sum_{n=1}^{\infty} n^k \sum_{j=1}^{n} \mathbb{E}\big[\mathbb{1}_{\{N=n\}} \zeta_j^k\big].$$

Applying Hölder inequality we get

$$\begin{aligned} \mathbb{E}\big[S^k\big] &\leq \sum_{n=1}^{\infty} n^k \sum_{j=1}^{n} \mathbb{E}\big[\zeta_j^{k(1+\delta)}\big]^{\frac{1}{1+\delta}} \mathbb{P}[N=n]^{\frac{\delta}{1+\delta}} \\ &= \mathbb{E}\big[\zeta_1^{k(1+\delta)}\big]^{\frac{1}{1+\delta}} \sum_{n=1}^{\infty} n^{k+1} \mathbb{P}[N=n]^{\frac{\delta}{1+\delta}}. \end{aligned}$$

Now applying Chebyshev inequality we get

$$\mathbb{E}[S^k] \leq \mathbb{E}\big[\zeta_1^{k(1+\delta)}\big]^{\frac{1}{1+\delta}} \sum_{n=1}^{\infty} n^{k+1} \frac{\mathbb{E}[N^l]^{\frac{\delta}{1+\delta}}}{n^{\frac{l\delta}{1+\delta}}} = \mathbb{E}\big[\zeta_1^{k(1+\delta)}\big]^{\frac{1}{1+\delta}} \mathbb{E}[N^l]^{\frac{\delta}{1+\delta}} \sum_{n=1}^{\infty} \frac{1}{n^{\frac{l\delta}{1+\delta}-(k+1)}} < \infty.$$

□

For $u \in \mathbb{Z}^2$ define

$$C_K(u) := \{u(1), \dots, u(1)+K-1\} \times \{u(2)-K+1, \dots, u(2)\}.$$

We say that the box $C_K(u)$ is good if for all $v \in C_K(u)$, $W_v = 1$ and v is open.

Remark 2.1.1. *We point out that when $C_K(u)$ is good then there are no paths crossing it and touching either $(-\infty, u(1)] \times \{u(2)\}$ or $[u(1)+K, \infty) \times \{u(2)\}$.*

Consider $e_1 = (1,0)$ and define the following random variables

$$g_K^+(u) := \inf\{n \geq 1 : C_K(u+(n-1)Ke_1) \text{ is good}\}$$

and

$$g_K^-(u) := \inf\{n \geq 1 : C_K(u-(nK-1)e_1) \text{ is good}\}.$$

Therefore $C_K\big(u+Kg_K^+(u)e_1\big)$ is the first translation of $C_K(u)$ to the right of u by multiples of K that is good and $C_K\big(u-Kg_K^-(e_1)\big)$ is the first translation of $C_K(u)$ to the left of u by multiples of K that is good.

Our next result says that the number of paths in $\mathcal{X}$ starting before time t that cross a finite length interval at time t have finite absolute moment of any order.

Lemma 2.1.2. *Let us define $\mathcal{X}^{t^-}$ as the set of paths in $\mathcal{X}$ that start before or at time t and by $X^{t^-}(t)$ its values on time t. Then we have that*

$$\mathbb{E}\Big[\big|\mathcal{X}^{t^-}(t) \cap [a,b]\big|^k\Big] < \infty$$

for $a < b \in \mathbb{R}$ and $k \geq 1$.

Proof. We will assume that $t = a = 0$ and $b = 1$. The general case is analogous. For $j \in \mathbb{Z}$ let us define $\zeta_j := \inf\{n \geq 0 : \sum_{i=0}^{n} \mathbb{1}_{\{(j,-i) \text{ is open }\}} = K\}$ and the random region D as

$$D := \{v \in \mathbb{Z}^2 : -Kg_K^-((0,0)) \leq v(1) \leq 1 + Kg_K^+((0,0)) \text{ and } -\zeta_{v(1)} \leq v(2) \leq 0\}.$$

In next figure we show a possible face of D.

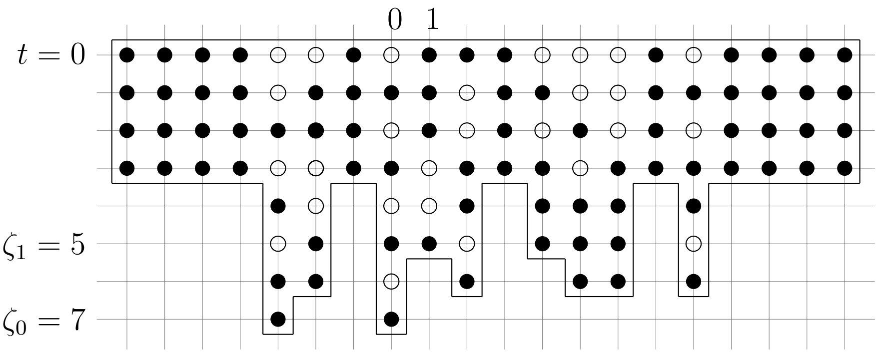

Figure 2.1: In this picture we assume that $K = 4$. The blacks balls represent open points and the white ones represent closed points. The region D is given by the set of sites inside the contour in bold. Note that $g_4^+ = 3$ and $g_4^- = 2$.

Recall that K is the support of W and note that, there are no paths of $\mathcal{X}$ crossing $[0,1] \times \{0\}$ without landing in D, hence

$$\left|\mathcal{X}^{0^-}(0) \cap [0,1]\right| \leq |D| = \sum_{j=1}^{Kg^+((1,0))} \zeta_j + \sum_{j=0}^{Kg^-((0,0))} \zeta_{-j}$$

Now using the Lemma 2.1.1 we have that $\mathbb{E}\left[\left|\mathcal{X}^{0^-}(0) \cap [0,1]\right|^k\right] < \infty$ for all $k \geq 1$. □

Proposition 2.1.1. *We have that $\bar{\mathcal{X}}_n$, the closure in $(\mathcal{H}, d_{\mathcal{H}})$ of $\mathcal{X}_n$, is a compact set of (Π, d) for all $n \geq 1$.*

Proof. We will write the proof for $\bar{\mathcal{X}}$ because for $\bar{\mathcal{X}}_n$ is analogous. As in [12] we will show that the mapping of $\mathcal{X}$ by $(\Phi(x,t), \Psi(t))$ - recall the definition of (Φ, Ψ) in (1.1.1) - is equicontinuous. Note that by the properties of $(\Phi(x,t), \Psi(t))$ it is enough to proof the collection of that paths touching a box $[-L, L] \times [L, L]$ is equicontinuous, for any $L > 0$. Using Lemma 2.1.2 we have that the number of paths in $\bar{\mathcal{X}}$ that cross $[-L, L] \times \{k\}$ is almost surely finite, then the number of paths touching a box $[-L, L] \times [L, L]$ is almost surely finite, and hence equicontinous. □

The Lemma below allow us to consider the counting variables $\eta_{\mathcal{X}_n}(t_0, t, a, b)$ only on integer starting times t_0.

Lemma 2.1.3. *Take $a, b, t_0, t \in \mathbb{R}$ with $a < b$ and $t > 0$, $\mathcal{X}_n$ as defined in (1.2.3), and $\eta_{\mathcal{X}_n}(t_0, t, a, b)$ as in Theorem 1.1.2. Then for all $\epsilon > 0$ there exists a constant M_ϵ, not depending on a, b, n, such that*

$$\mathbb{P}\big[|\eta_{\mathcal{X}_n}(t_0, t, a, b)| > 1\big] \leq \mathbb{P}\big[|\eta_{\mathcal{X}}(0, n^2\gamma t, n\sigma a - M_\epsilon, n\sigma b + M_\epsilon)| > 1\big] + \epsilon$$

for all $t_0 \in \mathbb{R}, t > 0$ and $n \geq 1$.

Proof. Note that any path that cross $[n\sigma a, n\sigma b] \times \{n^2\gamma t_0\}$ also cross the interval

$$\Big[n\sigma a - Kg_K^-\big((\lfloor n\sigma a\rfloor, \lfloor n^2\gamma t_0\rfloor + 1)\big), n\sigma b + Kg_K^+\big((\lfloor n\sigma b\rfloor + 1, \lfloor n^2\gamma t_0\rfloor + 1)\big)\Big]$$

at time $\lfloor n^2\gamma t_0\rfloor + 1$. Then

$$\begin{aligned}
&\mathbb{P}[|\eta_{\mathcal{X}_n}(t_0, t, a, b)| > 1] \\
&= \mathbb{P}\Big[|\eta_{\mathcal{X}}(n^2\gamma t_0, n^2\gamma t, n\sigma a, n\gamma b)| > 1] \\
&\leq \mathbb{P}\Big[\big|\eta_{\mathcal{X}}(\lfloor n^2\gamma t_0\rfloor), n^2\gamma t, n\sigma a - Kg^-\big((\lfloor n\sigma a\rfloor, \lfloor n^2\gamma t_0\rfloor + 1)\big), n\sigma b \\
&\quad + Kg^+\big((\lfloor n\sigma b\rfloor + 1, \lfloor n^2\gamma t_0\rfloor + 1)\big)\big| > 1\Big].
\end{aligned}$$

Take M_ϵ large enough such that

$$\mathbb{P}\Big[Kg^-\big((\lfloor n\sigma a\rfloor, \lfloor n^2\gamma t_0\rfloor + 1)\big) > M_\epsilon] = \mathbb{P}\Big[Kg^+\big((\lfloor n\sigma b\rfloor + 1, \lfloor n^2\gamma t_0\rfloor + 1)\big) > M_\epsilon\Big] \leq \frac{\epsilon}{2}.$$

Note that M_ϵ does not depend on a, nor b, neither n.

By the translation invariant we get

$$\begin{aligned}
\mathbb{P}[|\eta_{\mathcal{X}_n}(t_0, t, a, b)| > 1] &\leq \mathbb{P}\big[|\eta_{\mathcal{X}}(\lfloor n^2\gamma t_0\rfloor, n^2\gamma t, n\sigma a - M_\epsilon, n\sigma b + M_\epsilon)| > 1\big] + \epsilon \\
&= \mathbb{P}\big[|\eta_{\mathcal{X}}(0, n^2\gamma t, n\sigma a - M_\epsilon, n\sigma b + M_\epsilon)| > 1\big] + \epsilon.
\end{aligned}$$

□

2.2 Renewal times

As in [13] let us denote by $\Delta_k(u)$, for $k \in \mathbb{Z}_+$ and $u \in \mathbb{Z}^2$, the set of points above $X_k(u)$ whose configuration is already known; i.e. $\Delta_0(u) = \emptyset$ and for $k \geq 1$,

$$\begin{aligned}
\Delta_k(u) := &\Big[\Delta_{k-1}(u) \cup \big\{v \in \mathbb{Z}^2 : ||v - X_{k-1}(u)||_1 \leq ||X_k(u) - X_{k-1}(u)||_1\big\}\Big] \\
&\cap \big\{v \in \mathbb{Z}^2 : v(2) > X_k(u)(2)\big\}.
\end{aligned} \tag{2.2.1}$$

See Figure 2.2 as an example.

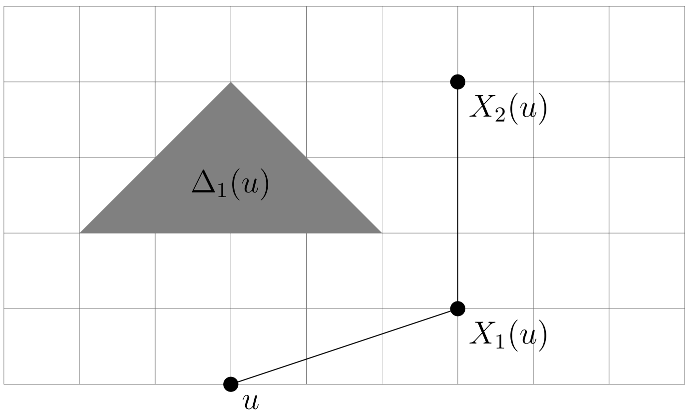

Figure 2.2: Example for the dependence region $\Delta_1(u)$. Note that in this example $\Delta_2(u) = \emptyset$.

For any random variable $\tau(u)$ which satisfies $\Delta_{\tau(u)}(u) = \emptyset$ we call $X_{\tau(u)}(u)(2)$ a renewal time for the random path $\{X_k(u); k \geq 1\}$. Note that $\tau(u)$ is not necessarily the first k such that $\Delta_k(u) = \emptyset$. The fact that the paths does not jump necessarily to first open level above it does not allow us to use the approach in [13] to verify existence and moment conditions of renewal times. The main result of this section is the following:

Proposition 2.2.1. *Let $u_1, \dots, u_m$ be points in $\mathbb{Z}^2$ at the same time level, i.e with equal second component. Then there exist random variables T, Z and $\tau(u_i)$ for $i = 1, \dots, m$ such that $T \leq Z$ and*

(i) $\Delta_{\tau(u_i)}(u_i) = \emptyset$ and $X_{\tau(u_1)}(u_1)(2) = X_{\tau(u_i)}(u_i)(2)$ for all $i = 1, \dots, m$.

(ii) Taking $T := X_{\tau(u_1)}(u_1)(2)$ we have that its distribution depends on m but not on $u_1, \dots, u_m$. For all $k \geq 1$ we get $\mathbb{E}[T^k] < \infty$. Note that $\pi^{u_i}(T) = X_{\tau(u_i)}(u_i)(1)$.

(iii) For all $i = 1, \dots, m$ we have $\sup_{0 \leq t \leq T} |\pi^{u_i}(t) - u_i(1)| \leq Z$ and its distribution depends on m but not on $u_1, \dots, u_m$. Also for all $k \geq 1$ we get $\mathbb{E}[Z^k] < \infty$.

Proof. Without lost of generality we can assume that $u_1(2) = \cdots = u_m(2) = 0$. For $u \in \mathbb{Z}^2$ let us define the following event $E(u)$ as

$$\big\{(u(1), u(2) + j) \text{ is open } : j = 1, \dots, K\big\} \cap \big\{W_{(u(1),u(2)+j)} = 1 : j = 1, \dots, K - 1\big\}.$$

Note that on $E(u)$ the path, that start in u and after some steps arrives in $(u(1), u(2)+K)$, knows nothing about the points above it. Now take $\{\widehat{E}_{1,1}, \widehat{E}_{1,1}, \dots, \widehat{E}_{1,m}\}$ independent events such that $\mathbb{P}[\widehat{E}_{1,j}] = \mathbb{P}[E((0,0))]$ for $j = 1, \dots, m$ and independent of the GRDF too. Let us define

$$D_1 := \{j \in \{1, \dots, m\} : u_j(1) = u_i(1) \text{ for some } 1 \leq i < j\}$$

and

$$E_1 := \bigcap_{j \in \{1,\dots,m\} \setminus D_1} E(u_j) \cap \big[\bigcap_{j \in D_1} \widehat{E}_{1,j}\big].$$

On the event E_1 the path that start at u_j, after some $\tau(u_j)$ steps, will arrive at $(u_j(1), u_j(2) + K)$ and $\Delta_{\tau(u_j)} = \emptyset$, for $j = 1, \dots, m$. We will find a sequence of independent events $\{E_n\}_{n \geq 1}$ with the same probability of success of E_1 such that if E_n occurs for some n, we have a joint renewal time for the paths that start in $u_1, \dots, u_m$. To do this let us make some definitions. Define the following upper bound for the height of $\Delta_1(u)$,

$$H(u) := \inf\Big\{n \geq 1 : \sum_{j=1}^{n} \mathbb{1}_{\{(u(1),u(2)+j) \text{ is open }\}} = K\Big\}. \tag{2.2.2}$$

Let $\{\widehat{H}_{1,j}; 1 \leq j \leq m\}$ be a collection of i.i.d. random variables, independent of the GRDF model, and with the same distribution as $H((0,0))$. Now define

$$\xi_1 := \Big[\max_{j \in \{1,\dots,m\} \setminus D_1} H(u_j)\Big] \vee \Big[\max_{j \in D_1} \widehat{H}_{1,j}\Big].$$

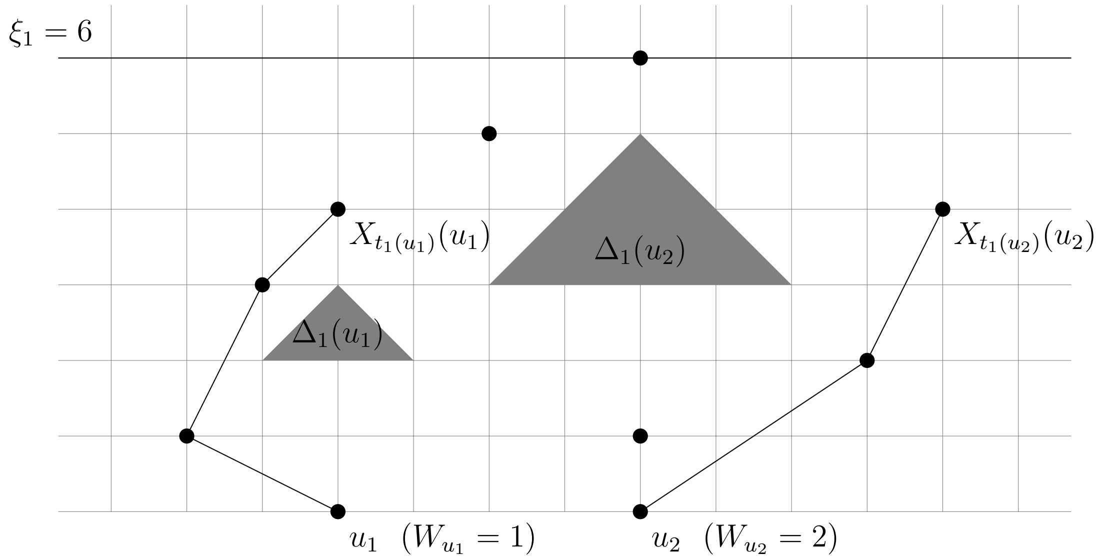

Figure 2.3: In the picture above we consider a realization of the random paths in the GRDF starting at u_1 and u_2. In this case $\xi_1 = 6$ and one can see that the dependence region generated by the first for both paths are below $u_1(2) + \xi_1$. Moreover note that $X_{t_1(u_1)}(u_1)$ and $X_{t_1(u_2)}(u_2)$ are the last points visited by the paths starting respectively at u_1 and u_2 before time $u_1(2) + \xi_1$.

Now we will move each path until the first time that they need to observe over ξ_1 to decide where to jump. These times could be defined as

$$t_1(u_j) := \inf\Big\{n \geq 1 : X_n(u_j)(2) = \xi_1 \text{ or } \sum_{i=1}^{\xi_1 - X_n(u_j)(2)} \mathbb{1}_{\{L(u_j,i) \text{ is open }\}} < W_{X_n(u_j)}\Big\},$$

for all $1 \leq j \leq m$. Note that to define $t_1(u_j)$ we do not need to see the points above ξ_1. To help the understanding of the notation see Figure 2.3.

Now take $\{\widehat{E}_{n,j}; 1 \leq j \leq m, n \geq 2\}$ a family of independent event and independent of the model too, such that $\mathbb{P}[\widehat{E}_{n,j}] = \mathbb{P}[E((0,0))]$ for all $1 \leq j \leq m$ and $n \geq 2$. Also take $\{\widehat{H}_{n,j}; 1 \leq j \leq m, n \geq 2\}$ an i.i.d. family of random variable independent of the model and with the same distribution of $H((0,0))$. Getting defined $E_1, \dots, E_m$ and $\xi_1, \dots, \xi_n$ we can define E_{n+1} and ξ_{n+1} as follows. First take $t_n(u_j)$ as

$$\inf\Big\{k \geq 1 : X_k(u_j)(2) = \xi_1 + \dots \xi_n \text{ or } W_{X_k(u_j)} > \sum_{i=1}^{\xi_1 + \dots \xi_k - X_k(u_j)(2)} \mathbb{1}_{\{L(u_j,i) \text{ is open }\}}\Big\},$$

for all $1 \leq j \leq m$. Define

$$D_{n+1} := \big\{j \in \{1, \dots, m\} : X_{t_n(u_j)}(u_j)(1) = X_{t_n(u_i)}(u_i)(1) \text{ for some } 1 \leq i < j\big\}$$

and

$$E_{n+1} := \Big[\bigcap_{i \in \{1,\dots,m\} \setminus D_{n+1}} E\big((X_{t_n(u_i)}(u_i)(1), \xi_1 + \dots + \xi_n)\big)\Big] \cap \Big[\bigcap_{j \in D_{n+1}} \widehat{E}_{n,j}\Big]$$

$$\xi_{n+1} := \Big[\max_{j \in \{1,\dots,m\} \setminus D_{n+1}} H\big((X_{t_n(u_j)}(u_j)(1), \xi_1 + \dots + \xi_n)\big)\Big] \vee \Big[\max_{j \in D_{n+1}} \widehat{H}_{n,j}\Big].$$

Note that $\{\xi_n\}_{n\geq 1}$ is an i.d.d. sequence and the distribution of ξ_n does not depend on $u_1(1),\dots,u_m(1)$. Also the probability of success of the event E_n does not depend on $u_1(1),\dots,u_m(1)$ and it is equal to $P[E_1]$. Notice that if E_n happens fore some n then we get that $\sum_{i=1}^{n}\xi_i$ is a renewal time for the paths. Now defining the geometric random variable $M := \inf\{n \geq 1 : \mathbb{1}_{E_n} = 1\}$ and $\tau(u_j) = t_M(u_j)$ we get $\Delta_{\tau(u_j)}(u_j) = \emptyset$ for all $1 \leq j \leq m$. Defining $T := \sum_{i=1}^{M}\xi_i$ we get that $T = X_{\tau(u_j)}(u_j)(2)$ for all $1 \leq j \leq m$ and applying Lemma 2.1.1, $\mathbb{E}[T^l] < \infty$ for all $l \in \mathbb{N}$. Note that the distribution of T does not depend on $u_1(1),\dots,u_m(1)$. Now see that for $j = 1\dots,m$ by the construction of $\{\xi_k; k \geq 1\}$ and $\{t_k(u_j); k \geq 1\}$ we have

$$\sup_{0\leq t\leq T} |\pi^{u_j}(t) - u_j(1)| \leq \sum_{k=1}^{M}(\xi_1 + \cdots + \xi_k)^2 \leq M(\xi_1 + \dots \xi_M)^2 \leq (\xi_1 + \dots \xi_M)^3 := Z.$$

Using Lemma 2.1.1 we get $\mathbb{E}[Z^l] < \infty$ for all $l \geq 1$ and, by construction, the distribution of Z does not depends on $u_1,\dots,u_m$.

□

We can replicate recursively Proposition 2.2.1 to get:

Corollary 2.2.1. *Let $m \geq 1$ and $u_1,\dots,u_m$ be points in $\mathbb{Z}^2$ at the same level. Then there exist sequences of random variables $\{T_j\}_{j\geq 1}$, $\{Z_j\}_{j\geq 1}$ and $\{\tau_j(u_i)\}_{j\geq 1}$ for $i = 1,\dots m$ such that,*

(i) $\Delta_{\tau_j(u_i)}(u_i) = \emptyset$ and $X_{\tau_j(u_1)}(u_1)(2) = X_{\tau_j(u_i)}(u_i)(2)$ for all $i = 1,\dots,m$ and $j \geq 1$. Moreover for every $i = 1,\dots m$ we have that $\tau_j(u_i) - \tau_{j-1}(u_i)$, $j \geq 1$, are i.i.d random variables.

(ii) $X_{\tau_j(u_1)}(u_1)(2) = T_j$ for $j \geq 1$ and its distribution depends on m but not on u_1 ,..., u_m. For all $k, j \geq 1$ we get that $\mathbb{E}[T_j^k] < \infty$ and for all $j \geq 1, i = 1\dots m$ we have that $\pi^{u_i}(T_j) = X_{\tau_j(u_i)}(u_i)(1)$. Moreover $T_j - T_{j-1}$, $j \geq 1$, are i.i.d random variables.

(iii) $\sup_{T_{j-1}\leq t\leq T_j} |\pi^{u_i}(t) - \pi^{u_i}(T_{j-1})| \leq Z_j$ for all $i = 1,\dots,m$ and $j \geq 1$. The random variables Z_j, $j \geq 1$, are i.i.d and their common distribution depends on m but not on $u_1,\dots,u_m$. For all $k, j \geq 1$ we get that $\mathbb{E}[Z_j^k] < \infty$.

Fix points $u_1,\dots,u_m$ in $\mathbb{Z}^2$. To simplify, suppose that u_1 is at the same time level or above u_i for every $i = 2,\dots m$. Then we can obtain a similar renewal structure for paths in $\mathcal{X}$, that do not start necessarily at the same time level, moving each path $\pi^{u_2},\dots,\pi^{u_m}$ up to the first time they need to see above $u_1(2)$ to move, and after that use the same idea of Proposition 2.2.1.

Corollary 2.2.2. *Let $m \geq 1$ and $u_1,\dots,u_m$ be points in $\mathbb{Z}^2$ such that u_1 is at the same time level or above u_i for every $i = 2,\dots m$. Then there exist sequences of random variables $\{T_j\}_{j\geq 1}$, $\{Z_j\}_{j\geq 1}$ and $\{\tau_j(u_i)\}_{j\geq 1}$ for $i = 1,\dots m$ such that,*

(i) $\Delta_{\tau_j(u_1)}(u_1) = \Delta_{\tau_j(u_i)}(u_i) = \emptyset$ and $X_{\tau_j(u_1)}(u_1)(2) = X_{\tau_j(u_i)}(u_i)(2)$ for all $i = 1,\dots,m$ and $j \geq 1$. Moreover we have that $\tau_j(u_1) - \tau_{j-1}(u_1)$, $j \geq 1$, are i.i.d random variables, and for every $i = 2,\dots m$ we have $\tau_j(u_i) - \tau_{j-1}(u_i)$, $j \geq 2$, are i.i.d random variables.

(ii) $X_{\tau_j(u_i)}(u_i)(2) = T_j$ *for all* $i = 1, \dots m$ *and* $j \geq 1$*. Also for all* $j \geq 1, i = 1 \dots m$ *we have that* $\pi^{u_i}(T_j) = X_{\tau_j(u_i)}(u_i)(1)$*. The distribution of* T_j *depends on* m *but not on* $u_1, \dots, u_m$ *and for all* $k, j \geq 1$ *we get that* $\mathbb{E}[T_j^k] < \infty$*. Moreover* $T_j - T_{j-1}$*,* $j \geq 2$*, are i.i.d random variables.*

(iii) $\sup_{T_{j-1} \leq t \leq T_j} |\pi^{u_1}(t) - \pi^{u_1}(T_{j-1})| \leq Z_j$ *for all* $j \geq 1$*, and for every* $i = 2, \dots m$ *we have that* $\sup_{T_{j-1} \leq t \leq T_j} |\pi^{u_i}(t) - \pi^{u_i}(T_{j-1})| \leq Z_j$ *for all* $j \geq 2$*. The random variables* Z_j*,* $j \geq 1$*, are i.i.d and their common distribution depends on* m *but not on* $u_1, \dots, u_m$*. For all* $k, j \geq 1$ *we get that* $\mathbb{E}[Z_j^k] < \infty$*.*

2.3 Coalescing time

The aim of this section is to prove the following Proposition 2.3.1, extremely useful to prove the conditions of Theorem 1.1.2.

Proposition 2.3.1. *Let us define* $\nu := \inf\{t \geq 0 : \pi^{(0,0)}(s) = \pi^{(0,1)}(s) \text{ for all } s \geq t\}$*. Then there exist a positive constant* $C > 0$ *such that*

$$\mathbb{P}[\nu > k] \leq \frac{C}{\sqrt{k}}.$$

As an immediate consequence of Proposition 2.3.1 we have:

Corollary 2.3.1. *Let* $u = (0, 0), v = (0, l)$ *and*

$$\nu(u, v) := \inf\{t \geq 0 : \pi^u(s) = \pi^v(s) \text{ for all } s \geq t\}.$$

Then there exist positive constant C *such that*

$$\mathbb{P}[\nu(u, v) > k] \leq \frac{Cl}{\sqrt{k}}.$$

Proof. Since $\{\nu(u, v) > k\} \subset \cup_{i=1}^{l} \{\nu\big((i-1)e_1, ie_1\big) > k\}$ we have that

$$\mathbb{P}[\nu(u, v) > k] \leq \sum_{i=1}^{l} \mathbb{P}\Big[\nu\big((i-1)e_1, ie_1\big) > k\Big] \leq \frac{Cl}{\sqrt{k}}. \tag{2.3.1}$$

□

We will need some previous results to prove Proposition 2.3.1. First, we are going to construct another sequence of renewals times $(\widehat{T}_i)_{i \geq 1}$ for two paths. The ideas used to get theses new times, are the same one used in Section 2.2.1, hence some details will be omitted. We will see later why in this section we need to consider this new renewal structure.

Let u and v two points in $\mathbb{Z} \times \{0\}$. Let $(T_i^u)_{i \geq 1}$ be the renewals for the path π^u that was getting in Corollary 2.2.1. Now move the path π^v up to the first time that they need to see above $T_1^u - K$, and see if the points between $T_1^u - K$ and T_1^u, in the same $y-$axis of the point where you stopped the path π^v, are open and with the respectively random variables W's, equals to one. If this happens, we have that in T_1^u, the paths π^u and π^v renews together. If not, we move the path π^v, up to the first time to need to see above

$T_2^u - K$ t, and again, see the appropriated K points between $[T_2^u - K, T_2^u]$ and ask the same question. Repeat this procedure until we get a "yes". By construction, we know nothing about the points between $T_j^u - K$ and T_j^u, for any $j \geq 1$,unless the ones associated to the path π^u. Then we get a common renewal time

$$\widehat{T} := \sum_{i=1}^{\widehat{G}} T_i^u$$

where $\widehat{G}$ is a geometric random variable. See that $\widehat{G}$ does not depend on the displacement of π^v, hence $\widehat{T}$ does not depend. Also the law of $\widehat{T}$ does not depend on v. Now define the random variable

$$\widehat{Z} := \sum_{i=1}^{\widehat{G}} (T_i)^2.$$

Then we have that

$$|\pi^u(\widehat{T}) - \pi^v(\widehat{T})| \leq 2\widehat{Z}.$$

In view of $\widehat{G}$ does not depend on the displacement of π^u, for all $m \in \mathbb{N}$, taking $u = (0,0)$ and $v = (0, m)$, we have that

$$\pi^{(0,0)}(\widehat{T}) - \pi^{(m,0)}(\widehat{T})|\{\pi^{(0,0)}(\widehat{T}) - \pi^{(m,0)}(\widehat{T}) < 0\} \overset{st}{\geq} -2\widehat{Z}. \tag{2.3.2}$$

Repeating this construction we get a sequence of common renewals times $(\widehat{T}_i)_{i\geq 1}$ and i.i.d. random variables $(\widehat{Z}_i)_{i\geq 1}$.

Let be $u_0 := (0,0)$ and for $m \in \mathbb{N}$ take $u_m := (m, 0)$ define

$$Y_0^m := m \text{ and } Y_n^m := \pi_n^{u_m}(\widehat{T}_n) - \pi_n^{u_0}(\widehat{T}_n) \text{ for } n \geq 1. \tag{2.3.3}$$

The process Y^m represents the distance between the paths $\pi^{(0,0)}$ and $\pi^{(0,m)}$ on the renewal times for the pair $(\pi^{(0,0)}, \pi^{(0,m)})$.

By the Skorohood's Representation Theorem, there exist a Brownian motion $(B(s))_{s\geq 0}$ starting in m and stopping times $(S_i)_{i\geq 0}$ such that

$$B(S_i) \overset{d}{=} Y_i, \text{ for } i \geq 0$$

and $(S_i)_{i\geq 0}$ has the following representation:

$$S_0 := 0,\ S_i := \inf\Big\{s \geq S_{i-1} : B(s) - B(S_{i-1}) \notin \big(U_i(B(S_{i-1})), V_i(B(S_{i-1}))\big)\Big\}$$

where $\big\{(U_i(m), V_i(m)) : m \in \mathbb{Z}, i \geq 1\big\}$ is a family of independent random vectors taking values in $\big((\mathbb{Z}_- - \{0\}) \times \mathbb{N}\big) \cup \big\{(0,0)\big\}$.

Lemma 2.3.1. *For any borel set $\mathbb{A}$, let us define $\nu_{\mathbb{A}}^m$ as the first hitting time to $\mathbb{A}$ of the sequence $\{Y_n^m\}_{n\geq 1}$; i.e.*

$$\nu_{\mathbb{A}}^m := \inf\{n \geq 1 : Y_n^m \in \mathbb{A}\}.$$

Then

(i) For all $m \in \mathbb{N}$ we have $\mathbb{P}\big[\nu^m_{(-\infty,0]} < \infty\big] = 1$.

(ii) $\sup_{m<0} \mathbb{E}\big[Y^m_{\upsilon^m_{[0,\infty)}}\big] < \infty$.

(iii) $\inf_{m \geq 1} \mathbb{P}\big[Y^m_{\nu^m_{(-\infty,0]}} = 0\big] > 0$.

(iv) Let us define the sequence $(a_l)_{l \geq 1}$ as

$$a_1 := \inf\{n \geq 1 : Y^1_n \leq 0\} \text{ and for } l \geq 2,$$

and

$$a_l := \begin{cases} \inf\{n \geq a_{l-1} : Y^1_n \geq 0\}; & \text{if } l \text{ is even} \\ \inf\{n \geq a_{l-1} : Y^1_n \leq 0\}; & \text{if } l \text{ is odd} \end{cases}.$$

Then there exists a constant $c_1 < 1$ such that

$$\mathbb{P}[Y^1_{a_j} \neq 0, \text{ for } j = 1, \dots, k] \leq c_1^k,$$

for all $k \geq 1$.

Proof. Let us start proving item (i). Define

$$D := \big\{n \in [1, \nu^m_{(-\infty,0]}] \cap \mathbb{N} : (B(s))_{s \geq 0} \text{ visits } (-\infty, 0] \text{ in the interval } (S_{n-1}, S_n]\big\}$$

Since $m > 0$, for any $n \in D$, one of the following things should occur:

(a) $U_n(B(S_{n-1})) + B(S_{n-1}) = 0$. This implies that $n = \nu^m_{(-\infty,0]}$, since $B(S_n) = B(S_{n-1}) + U_n(B(S_{n-1}))$.

(b) $U_n(B(S_{n-1})) + B(S_{n-1}) < 0$. In this case we have that $(B(s))_{s \geq 0}$ visits zero in the interval $(S_{n-1}, S_n]$, before hitting $\{V_n(B(S_{n-1})) + B(S_{n-1}), B_n(B(S_{n-1})) + B(S_{n-1})\}$. Since $B(S_n) - B(S_{n-1}) \overset{d}{=} Y_n - Y_{n-1}$, we get - using Strong Markov property and (2.3.2) - that the probability of an Standard Brownian motion, independent of $\widehat{Z}$, leaves the interval $[-2\widehat{Z}, 1]$ by the left side, is a lower bound to the probability that $B(S_n) = U(B(S_{n-1})) + B(S_{n-1})$.

Hence $\#D \leq G$, where G is a Geometric random variable. Then $\#D < \infty$ almost surely. Let us see this implies $\nu^m_{(-\infty,0]} < \infty$ almost surely. If $\nu^m_{(-\infty,0]} = \infty$ and $\#D < \infty$, taking $j = \max\{n \in D\}$, we have that $(B(s))_{s \geq 0}$ does not visits zero after S_j; what happens with probability zero.

Let us prove item (ii). By item (i) we have that for all $m \in \mathbb{N}$

$$\mathbb{E}\big[Y^m_{\upsilon^m_{[0,\infty)}}\big] \leq \mathbb{E}\Big[\sum_{i=1}^{G} 2\widehat{Z}_i\Big] < \infty.$$

To prove (iii) consider $m, M \in \mathbb{N}$ with $m > M$ and define the stopping time

$$\nu^m_{(-\infty,M]} := \inf\{n \geq 1 : Y^m_n \leq M\}.$$

It is easy to see that $\nu^m_{(-\infty,M]} \leq \nu^m_{(-\infty,0]}$, hence by the item (i) we get that $\nu^m_{(-\infty,M]}$ is finite almost surely. Also note that

$$\begin{aligned}
\mathbb{P}[Y^m_{\nu^m_{(-\infty,0]}} = 0] &\geq \mathbb{P}[Y^m_{\nu^m_{(-\infty,0]}} = 0, \nu^m_{(-\infty,0]} \neq \nu^m_{(-\infty,M]}] \\
&= \sum_{k=1}^{M} \mathbb{P}[Y^m_{\nu^m_{(-\infty,0]}} = 0, Y^m_{\nu^m_{(-\infty,M]}} = k] \\
&= \sum_{k=1}^{M} \mathbb{P}[Y^m_{\nu^m_{(-\infty,0]}} = 0 | Y^m_{\nu^m_{(-\infty,M]}} = k]\mathbb{P}[Y^m_{\nu^m_{(-\infty,M]}} = k].
\end{aligned}$$

For all $1 \leq k \leq M$ by Strong Markov property of $(Y^m_n)_{n\geq 0}$ and the translation invariant of the model, we have that

$$\mathbb{P}[Y^m_{\nu^m_{(-\infty,0]}} = 0 | Y^m_{\nu^m_{(-\infty,M]}} = k] = \mathbb{P}[Y^k_{\nu^k_{(-\infty,0]}} = 0].$$

Hence

$$\begin{aligned}
\mathbb{P}[Y^m_{\nu^m_{(-\infty,0]}} = 0] &\geq \sum_{k=1}^{M} \mathbb{P}[Y^k_{\nu^k_{(-\infty,0]}} = 0]\mathbb{P}[Y^m_{\nu^m_{(-\infty,M]}} = k] \\
&\geq \Big(\min_{1\leq k\leq M} \mathbb{P}[Y^k_{\nu^k_{(-\infty,0]}} = 0]\Big) \sum_{k=1}^{M} \mathbb{P}[Y^m_{\nu^m_{(-\infty,M]}} = k] \\
&= \Big(\min_{1\leq k\leq M} \mathbb{P}[Y^k_{\nu^k_{(-\infty,0]}} = 0]\Big)\mathbb{P}[\nu^m_{(-\infty,0]} \neq \nu^m_{(-\infty,M]}] \\
&\geq \Big(\min_{1\leq k\leq M} \mathbb{P}[Y^k_{\nu^k_{(-\infty,0]}} = 0]\Big)\Big(\inf_{\tilde{m}>M} \mathbb{P}[\nu^{\tilde{m}}_{(-\infty,0]} \neq \nu^{\tilde{m}}_{(-\infty,M]}]\Big).
\end{aligned}$$

From the description of the GRDF it is straightforward to verify that $\mathbb{P}[Y^k_{\nu^k_{(-\infty,0]}} = 0] > 0$ for all $k \geq 1$, and then, $\min_{1\leq k\leq M} \mathbb{P}[Y^k_{\nu^k_{(-\infty,0]}} = 0] > 0$. Now let us to prove that for an adequate M we have

$$\inf_{\tilde{m}>M} \mathbb{P}[\nu^{\tilde{m}}_{(-\infty,0]} \neq \nu^{\tilde{m}}_{(-\infty,M]}] > 0.$$

Note that $\mathbb{P}[\nu^{\tilde{m}}_{(-\infty,0]} = \nu^{\tilde{m}}_{(-\infty,M]}] = \mathbb{P}[Y^{\tilde{m}}_{\nu^{\tilde{m}}_{(-\infty,M]}} \leq 0]$. By the symmetry and the invariant translation property of the GRDF model, we have

$$\mathbb{P}[Y^{\tilde{m}}_{\nu^{\tilde{m}}_{(-\infty,M]}} \leq 0] = \mathbb{P}[Y^{(M-\tilde{m})}_{\nu^{(M-\tilde{m})}_{[0,\infty)}} \geq M] \leq \frac{1}{M}\mathbb{E}\big[Y^{(M-\tilde{m})}_{\nu^{(M-\tilde{m})}_{[0,\infty)}}\big] \leq \frac{1}{M}\sup_{m<0}\mathbb{E}\big[Y^m_{v^m_{[0,\infty)}}\big].$$

By item (ii) we have that $\sup_{m<0}\mathbb{E}\big[Y^m_{v^m_{[0,\infty)}}\big] < \infty$, hence taking M such that

$$c := \frac{1}{M}\sup_{m<0}\mathbb{E}\big[Y^m_{v^m_{[0,\infty)}}\big] < 1,$$

we get

$$\inf_{\tilde{m}>M} \mathbb{P}[Y^{\tilde{m}}_{\nu^{\tilde{m}}_{(-\infty,0]}} = 0] \geq \Big(\min_{1\leq k\leq M} \mathbb{P}[Y^k_{\nu^k_{(-\infty,0]}} = 0]\Big)(1 - \frac{c}{M}) > 0,$$

which completes the proof of (iii).

Now we prove (iv). Define

$$c_1 := \sup_{m\geq 1} \mathbb{P}[Y^m_{a_1} \neq 0].$$

By the item (iii) we have $c_1 < 1$. By definition $\mathbb{P}[Y^1_{a_1} \neq 0] \leq c_1$. The proof will follow by induction on k. Suppose that $\mathbb{P}[Y^1_{a_j} \neq 0, \text{ for } j = 1, \ldots, k] \leq c_1^k$. Here we are going to assume that k is even, the case k odd is similar. Write

$$\begin{aligned}
&\mathbb{P}[Y^1_{a_j} \neq 0 \text{ for } j = 1, \ldots, k+1] \\
&= \sum_{m\geq 1} \mathbb{P}[Y^1_{a_{k+1}} \neq 0, Y^1_{a_k} = m, Y^1_{a_j} \neq 0 \text{ for } j = 1, \ldots, k-1] \\
&= \sum_{m\geq 1} \mathbb{P}[Y^1_{a_{k+1}} \neq 0 | Y^1_{a_k} = m, \cap_{j=1}^{k-1}\{Y^1_{a_j} \neq 0\}]\mathbb{P}[Y^1_{a_k} = m, \cap_{j=1}^{k-1}\{Y^1_{a_j} \neq 0\}].
\end{aligned}$$

By Strong Markov property of $(Y^1_n)_{n\geq 0}$ and the translation invariant of the GRDF model, we have that

$$\mathbb{P}[Y^1_{a_{k+1}} \neq 0 | Y^1_{a_k} = m, Y^1_{a_j} \neq 0 \text{ for } j = 1, \ldots, k-1] = \mathbb{P}[Y^m_{a_1} \neq 0] \leq c_1.$$

Hence

$$\begin{aligned}
\mathbb{P}[Y^1_{a_j} \neq 0 \text{ for } j = 1, \ldots, k+1] &\leq c_1 \sum_{m\geq 1} \mathbb{P}[Y^1_{a_k} = m, Y^1_{a_j} \neq 0 \text{ for } j = 1, \ldots, k-1] \\
&= c_1 \mathbb{P}[Y^1_{a_j} \neq 0 \text{ for } j = 1, \ldots, k] \\
&\leq c_1^{k+1}.
\end{aligned}$$

□

Lemma 2.3.2. *Consider the sequence $(a_l)_{l\geq 1}$ as in the previous lemma and $(B(s))_{s\geq 0}$ and $(S_n)_{n\geq 1}$ obtained from the Skorohood's Representation of Y^1. Then there exist random variables $(R_i)_{i\geq 1}$ and $\widetilde{R}_0$ independent of $(B(s))_{s\geq 0}$ such that*

(i) $R_i | \{R_i \neq 0\} \stackrel{d}{=} \widetilde{R}_0$ *for all* $i \geq 1$.

(ii) S_{a_l} *is stochastically dominated by* J_l*, which is defined as* $J_0 = 0$,

$$J_1 := \inf\{s \geq 0 : B(s) - B(0) = -(R_1 + \widetilde{R}_0)\},$$

and

$$J_l := \inf\{s \geq J_{l-1} : B(s) - B(J_{l-1}) = (-1)^l (R_l + R_{l-1})\}, l \geq 2.$$

(iii) $Y_{a_l} \neq 0$ *implies* $B(J_l) \neq 0$.

Proof. The proof is similar to the proof of Lemma 3.5 in [6]. By Skorohood Representation Theorem we have a Brownian motion $(B(s))_{s\geq 0}$ starting in 1 and stopping times $(S_n)_{n\geq 0}$, which could be both taken independent of $(Y^1_n)_{n\geq 1}$, such that

$$Y^1_n - Y^1_{n-1} \overset{d}{=} B(S_n) - B(S_{n-1})\,, \text{ for all } n \geq 1.$$

There exists a sequence of random variables $(\widehat{Z}_n)_{n\geq 1}$ such that $|Y^1_n - Y^1_{n-1}| \leq 2\widehat{Z}_n$. Then

$$|B(S_n) - B(S_{n-1})| \overset{st}{\leq} 2\widehat{Z}_n\,, \text{ for all } n \geq 1. \tag{2.3.4}$$

Recall that $(S_i)_{i\geq 0}$ has the following representation.

$$S_0 := 0, S_i := \inf\Big\{s \geq S_{i-1} : B(s) - B(S_{i-1}) \notin \big(U_i(B(S_{i-1})), V_i(B(S_{i-1}))\big)\Big\} \tag{2.3.5}$$

where $\big\{(U_i(m), V_i(m)); m \in \mathbb{Z}, i \geq 1\big\}$ is a family of independent random vectors taking values in $\big((\mathbb{Z}_- - \{0\}) \times \mathbb{N}\big) \cup \big\{(0, 0)\big\}$.
By (2.3.4) and (2.3.5) we have that

$$-2\widehat{Z}_n \overset{st}{\leq} \inf_{S_{n-1}\leq s\leq S_n}\{B(s) - B(S_{n-1})\} \leq \sup_{S_{n-1}\leq s\leq S_n}\{B(s) - B(S_{n-1})\} \overset{st}{\leq} 2\widehat{Z}_n \tag{2.3.6}$$

As in the proof of Lemma 2.3.1 consider the following random set

$$C := \Big\{n \in [1, \tau_{(-\infty,0]}] \cap \mathbb{N} : (B(s))_{s\geq 0} \text{ visits } (-\infty, 0] \text{ in the interval } (S_{n-1}, S_n]\Big\}.$$

Note that

$$-\sum_{i=1}^{|C|} 2\widehat{Z}_i \overset{st}{\leq} \inf_{0\leq s\leq \tau_{(-\infty,0]}} B(s)\,.$$

There exists a geometric random variable G (see the proof of item (i) of Lemma 2.3.1) such that $G \geq |C|$. Then we have

$$-\sum_{i=1}^{G} 2\widehat{Z}_i \overset{st}{\leq} \inf_{0\leq s\leq \tau_{(-\infty,0]}} B(s).$$

Let us define the random variable R_1 as

$$R_1 := \begin{cases} -\sum_{i=1}^{G} 2\widehat{Z}_i; \text{ if } |C| > 1 \\ 0; \text{ otherwise,} \end{cases}$$

and $\widetilde{R}_0$ as an independent random variable such that $\widetilde{R}_0 \overset{d}{=} R_1|\{R_1 \neq 0\}$. Define

$$J_1 := \inf\{s \geq 0 : B(s) - B(0) = -(R_1 + \widetilde{R}_0)\}$$

which is clearly stochastically above S_{a_1}. Let $(\mathbb{B}(s))_{s\geq 0}$ be $(B(s))_{s\geq 0}$ translated to have $\mathbb{B}(0) = \widetilde{R}_0$, then $Y_{a_1} \neq 0$ is equivalent to $B(J_1) \neq 0$, indeed

$$Y_{a_1} \neq 0 \Leftrightarrow R_1 > 0 \Leftrightarrow \mathbb{B}(J_1) < 0\,.$$

From this point, it is straightforward to use an induction argument to build the sequence $\{R_j\}_{j\geq 1}$. At step j in the induction argument, we consider initially an excursion of $(\mathbb{B}(s))_{s\geq 0}$ in a time interval of size $(S_{a_j} - S_{a_{j-1}})$, and since $|Y^n_{a_{j-1}}| \leq R_{j-1}$ we can obtain R_j and define J_j using $(\mathbb{B}(s))_{s\geq 0}$ as before. By the strong Markov property of (Y^1_n), we obtain that the R_j's are independent and $Y_{a_j} \neq 0$ is equivalent to $B(J_j) \neq 0$. □

Now consider $Y_n := Y_n^1$ for all $n \geq 1$ and take $\nu^Y := \inf\{n \geq 1 : Y_n^1 = 0\}$. We will obtain the Proposition 2.3.1 from the following lemma:

Lemma 2.3.3. *There exist positive constants C_1 and C_2 such that*

$$\mathbb{P}[\nu^Y > k] \leq \frac{C_1}{\sqrt{k}} \tag{2.3.7}$$

and

$$\mathbb{P}[T_{\nu^Y} > k] \leq \frac{C_2}{\sqrt{k}} \text{ for all } k \geq 1. \tag{2.3.8}$$

for every $k \geq 1$.

Proof. Let us suppose that (2.3.7) is true and use it to prove (2.3.8) with the same idea used in [13]. Recall from Corollary 2.2.1 that T_1 has finite moments and define the constant $L := 1/2\mathbb{E}[T_1]$ and take $k \in \mathbb{N}$ then

$$\mathbb{P}[T_{\nu^Y} > k] \leq \mathbb{P}[T_{\nu^Y} > k, \nu^Y \leq Lk] + \mathbb{P}[\nu^Y > Lk] \leq \mathbb{P}[T_{\lfloor Lk \rfloor} > k] + \mathbb{P}[\nu^Y > Lk].$$

By (2.3.7), it is enough to prove that $\mathbb{P}[T_{\lfloor Lk \rfloor} > k] \leq \frac{C_3}{\sqrt{k}}$ for some constant C_3. Then

$$\begin{aligned}
\mathbb{P}\big[T_{\lfloor Lk \rfloor} > k\big] &= \mathbb{P}\Big[\sum_{i=1}^{\lfloor Lk \rfloor}[T_i - T_{i-1}] > k\Big] \\
&= \mathbb{P}\Big[\sum_{i=1}^{\lfloor Lk \rfloor}[T_i - T_{i-1}] - \lfloor Lk \rfloor \mathbb{E}[T_1] > k - \lfloor Lk \rfloor \mathbb{E}[T_1]\Big] \\
&\leq \frac{\mathbb{V}ar\Big[\sum_{i=1}^{\lfloor Lk \rfloor}(T_i - T_{i-1})\Big]}{\Big(k - \lfloor Lk \rfloor \mathbb{E}[T_1]\Big)^2} \\
&= \frac{\lfloor Lk \rfloor \mathbb{V}ar[T_1]}{\Big(k - \lfloor Lk \rfloor \mathbb{E}[T_1]\Big)^2}.
\end{aligned}$$

Note that

$$\sqrt{k}\frac{\lfloor Lk \rfloor \mathbb{V}ar[T_1]}{\Big(k - \lfloor Lk \rfloor \mathbb{E}[T_1]\Big)^2} \to 0 \text{ as } k \to 0.$$

Then there exist M such that

$$\frac{\lfloor Lk \rfloor \mathbb{V}ar[T_1]}{\Big(k - \lfloor Lk \rfloor \mathbb{E}[T_1]\Big)^2} \leq \frac{1}{\sqrt{k}}$$

for all $k \geq M$. Then we can find a sufficiently large constant $C_3 > 0$ such that

$$\frac{\lfloor Lk \rfloor \mathbb{V}ar[T_1]}{\Big(k - \lfloor Lk \rfloor \mathbb{E}[T_1]\Big)^2} \leq \frac{C_3}{\sqrt{k}}$$

for all $k \geq 1$.

Now we prove (2.3.7). Here we simply write $Y = Y^1$. Recall the definition of $(B(s))_{s\geq 0}$ and $(S_i)_{i\geq 0}$ above for the case $m = 1$. For $\delta > 0$ to be fixed later and every $k \in \mathbb{N}$ we have that,

$$\mathbb{P}[\nu^Y > k] = \mathbb{P}[S_k \leq \delta k, \nu^Y > k] + \mathbb{P}[S_k > \delta k, \nu^Y > k]. \tag{2.3.9}$$

To get an upper bound for $\mathbb{P}[S_k \leq \delta k, \nu^Y > k]$ we use the Skorohod representation:

$$S_k = \sum_{i=1}^{k} \big(S_i - S_{i-1}\big) = \sum_{i=1}^{k} Q_i(Y_{i-1}),$$

where $\{Q_i(m); i \geq 1, m \in \mathbb{Z}\}$ are independent random variables and $Q_i(m)$ is independent of $(Y_1, ..., Y_{i-1})$ for all $i \in \mathbb{N}, m \in \mathbb{Z}$. Note that on $\{\nu^Y > k\}$ we have that $Y_i \neq 0$ for $i \in \{1, \ldots, k\}$. Now, for fixed $\lambda > 0$, we have

$$\mathbb{P}[S_k \leq \delta k, \nu^Y > k] = \mathbb{P}[e^{-\lambda S_k} \geq e^{-\lambda \delta k}, \nu^Y > k] \leq e^{\lambda \delta k}\mathbb{E}[e^{-\lambda S_k}\mathbb{1}_{\{\nu^Y > k\}}]$$

Claim 2.3.1.

$$\mathbb{E}[e^{-\lambda S_k}\mathbb{1}_{\{\nu^Y > k\}}] \leq \Big(\sup_{m\in\mathbb{Z}\setminus\{0\}} \mathbb{E}[e^{-\lambda Q(m)}]\Big)^k,$$

where, for each m, $Q(m)$ is a random variable with the same distribution of $Q_1(m)$.

Proof of Claim 2.3.1. The proof is essentially the same given in Theorem 4 in [5] and we include it here for the sake of completeness. Taking $\mathcal{F}_k := \sigma(Y_1, \ldots, Y_k)$ we have that

$$\begin{aligned}
\mathbb{E}[e^{-\lambda S_k}\mathbb{1}_{\{\nu^Y > k\}}] &= \mathbb{E}\Big[\mathbb{E}[e^{-\lambda S_k}\mathbb{1}_{\{\nu^Y > k\}}|\mathcal{F}_{k-1}]\Big] \\
&\leq \mathbb{E}\Big[e^{-\lambda S_{k-1}}\mathbb{E}[e^{-\lambda Q_k(Y_{k-1})}\mathbb{1}_{\{\nu^Y > k-1\}}\mathbb{1}_{\{Y_{k-1}\neq 0\}}|\mathcal{F}_{k-1}]\Big] \\
&= \mathbb{E}\Big[e^{-\lambda S_{k-1}}\mathbb{1}_{\{\nu^Y > k-1\}}\mathbb{E}[e^{-\lambda Q_k(Y_{k-1})}\mathbb{1}_{\{Y_{k-1}\neq 0\}}|\mathcal{F}_{k-1}]\Big]
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}[e^{-\lambda Q_k(Y_{k-1})}\mathbb{1}_{\{Y_{k-1}\neq 0\}}|\mathcal{F}_{k-1}] &= \sum_{m\in\mathbb{Z}\setminus\{0\}} \mathbb{E}[e^{-\lambda Q_k(m)}\mathbb{1}_{\{Y_{k-1}=m\}}|\mathcal{F}_{k-1}] \\
&= \sum_{m\in\mathbb{Z}\setminus\{0\}} \mathbb{1}_{\{Y_{k-1}=m\}}\mathbb{E}[e^{-\lambda Q_k(m)}|\mathcal{F}_{k-1}] \\
&= \sum_{m\in\mathbb{Z}\setminus\{0\}} \mathbb{1}_{\{Y_{k-1}=m\}}\mathbb{E}[e^{-\lambda Q(m)}] \\
&\leq \sup_{m\in\mathbb{Z}\setminus\{0\}} \mathbb{E}[e^{-\lambda Q(m)}].
\end{aligned}$$

So, applying the above argument recursively we obtain

$$\mathbb{E}[e^{-\lambda S_k}\mathbb{1}_{\{\nu^Y > k\}}] \leq \mathbb{E}[e^{-\lambda S_{k-1}}\mathbb{1}_{\{\nu^Y > k-1\}}]\Big(\sup_{m\in\mathbb{Z}\setminus\{0\}} \mathbb{E}[e^{-\lambda Q(m)}]\Big) \leq \Big(\sup_{m\in\mathbb{Z}\setminus\{0\}} \mathbb{E}[e^{-\lambda Q(m)}]\Big)^k.$$

□

Using Claim 2.3.1 we get that

$$\mathbb{P}[S_k \le \delta k, \nu^Y > k] \le \Big(e^{\lambda\delta} \sup_{m\in\mathbb{Z}\setminus\{0\}} \mathbb{E}[e^{-\lambda Q(m)}]\Big)^k .$$

Let $Q_{-1,1}$ be the exit time of interval $(-1,1)$ by a Standard Brownian motion. We have that

$$\begin{aligned}\mathbb{E}\big[e^{-\lambda Q(m)}\big] &= \mathbb{E}\big[e^{-\lambda Q(m)}|(U(m),V(m)) \neq (0,0))\big]\mathbb{P}[(U(m),V(m)) \neq (0,0)] \\ &\quad + \mathbb{P}\big[(U(m),V(m)) = (0,0)\big] \\ &\le \mathbb{E}\big[e^{-\lambda Q_{-1,1}}\big]\Big(1 - \mathbb{P}\big[(U(m),V(m)) = (0,0)\big]\Big) + \mathbb{P}\big[(U(m),V(m)) = (0,0)\big] \\ &= \mathbb{P}\big[(U(m),V(m)) = (0,0)\big](1-c_2) + c_2\,, \end{aligned} \tag{2.3.10}$$

where $c_2 = \mathbb{E}\big[e^{-\lambda Q_{-1,1}}\big] < 1$.

Claim 2.3.2. $0 < c_3 := \sup_{m\in\mathbb{Z}\setminus\{0\}} \mathbb{P}\big[(U(m),V(m)) = (0,0)\big] < 1$.

Proof of Claim 2.3.2. The proof uses the hypothesis that $P(W=1) > 0$. However by a straightforward adaptation, one can see that this is not required for the claim to remain valid.

We follow the idea used in [13]. Notice that

$$\mathbb{P}[\big(U(m),V(m)\big) = (0,0)] = \mathbb{P}[Y_{n+1} = m | Y_n = m].$$

Then (see Figure 2.4) we have

$$\mathbb{P}[\big(U(m),V(m)\big) = (0,0)] \ge (p\mathbb{P}[W=1])^2.$$

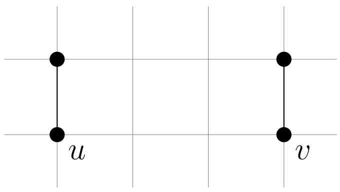

Figure 2.4: If $W_u = W_v = 1$ and $u+e_2$ and $v+e^2$ are open, which occurs with probability $(p\mathbb{P}[W=1])^2$.

So $c_3 := \sup_{m\in\mathbb{Z}\setminus\{0\}} \Big\{\mathbb{P}[\big(U(m),V(m)\big) = (0,0)]\Big\} \ge (p\mathbb{P}[W=1])^2 > 0$.

For the upper bound in the statement, see that

$$\mathbb{P}[\big(U(m),V(m)\big) \neq (0,0)] = \mathbb{P}[Y_{n+1} \neq m | Y_n = m].$$

Hence we have that (see Figure 2.5)

$$\mathbb{P}[\big(U(m),V(m)\big) \neq (0,0)] \ge (1-\frac{p}{2})p^3(1-p)^3(\mathbb{P}[W=1])^3,$$

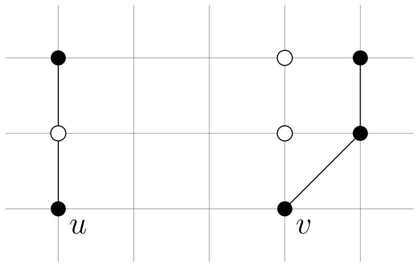

Figure 2.5: If $W_u = W_v = W_{v+e_1+e_2} = 1$ and $u+2e_2$, $v+e_1+e_2$ and $v+e_1+2e_2$ are open and $u+e_2$, $v+e_2$ and $v+2e_2$ are closed, then with $p^3(1-p)^4(\mathbb{P}[W=1])^3$ if $|m|=1$ and $(1-\frac{p}{2})p^3(1-p)^3(\mathbb{P}[W=1])^3$ in case $|m|\neq 1$, we have the trajectory as represented in the picture.

Hence $\inf_{m\in\mathbb{Z}\setminus\{0\}} \mathbb{P}[\big(U(m),V(m)\big)\neq(0,0)] \geq (1-\frac{p}{2})p^3(1-p)^3(\mathbb{P}[W=1])^3$. So,

$$c_3 \leq 1-(1-\frac{p}{2})p^3(1-p)^3(\mathbb{P}[W=1])^3 < 1.$$

□

Using Claim 2.3.2 and (2.3.10), we obtain that

$$\mathbb{E}\big[e^{-\lambda Q(m)}\big] \leq c_3(1-c_2)+c_2\,.$$

Now chose δ such that $c_4 := e^{\delta\lambda}\big[c_3(1-c_2)+c_2\big] < 1$. Then

$$\mathbb{P}[S_k \leq \delta k, \nu^Y > k] \leq c_4^k \leq \frac{c_5}{\sqrt{k}}, \tag{2.3.11}$$

for some suitable $c_5>0$. This gives the bound we need on the first term of (2.3.9). Let just prove the previous claim before dealing with the second term in (2.3.9).

Now we consider the second term in (2.3.9). To deal with it we consider an approach similar to [6]. Take the sequence $(a_l)_{l\geq 0}$ as in the statement of Lemma 2.3.1. Note that

$$\mathbb{P}[\nu^Y > k, S_k > \delta k] \leq \sum_{l=1}^{k} \mathbb{P}[\nu^Y > k, S_k \geq \delta k, S_{a_{l-1}} < \delta k, S_{a_l} \geq \delta k]\,. \tag{2.3.12}$$

For now fix $l=1,\dots k$, using Lemma 2.3.2 we get,

$$\{\nu^Y > k, S_k \geq \delta k, S_{a_{l-1}} < \delta k, S_{a_l} \geq k\delta\} \subseteq \{Y_{a_j}\neq 0, \text{ for } j=1,\dots,l-1, S_{a_l}\geq k\delta\},$$

hence

$$\begin{aligned}
&\mathbb{P}[\nu^Y > k, S_k \geq \delta k, S_{a_{l-1}} < \delta k, S_{a_l} \geq k\delta] \\
&\leq \mathbb{P}[Y_{a_j}\neq 0, \text{ for } j=1,\dots,l-1, S_{a_l}\geq k\delta] \\
&\leq \mathbb{P}[Y_{a_j}\neq 0, \text{ for } j=1,\dots l-1, J_l\geq k\delta] \\
&= \mathbb{P}[J_l\geq k\delta | Y_{a_j}\neq 0, \text{ for } j=1,\dots l-1]\mathbb{P}[Y_{a_j}\neq 0, \text{ for } j=1,\dots l-1]
\end{aligned}$$

By the item (iii) in the Lemma 2.3.1 we get

$$\mathbb{P}[\nu^Y > k, S_k \geq \delta k, S_{a_{l-1}} < \delta k, S_{a_l} \geq k\delta] \leq c_1^{l-1}\mathbb{P}[J_l\geq k\delta | Y_{a_j}\neq 0, \text{ for } j=1,\dots l-1].$$

Now take $(\widetilde{R}_j)_{j\geq 1}$ i.i.d. random variables independent of $(B(s))_{s\geq 0}$ such that $\widetilde{R}_1 =^d R_0$ and define $\widetilde{J}_0 = 0$,

$$\widetilde{J}_j := \inf\{s \geq \widetilde{J}_{j-1} : B(s) - B(\widetilde{J}_{j-1}) = (-1)^l(\widetilde{R}_l + \widetilde{R}_{l-1})\}, j \geq 1.$$

We have that

$$\mathbb{P}[J_j \geq k\delta | Y_{a_j} \neq 0, \text{ for } j = 1, \dots l-1] = \mathbb{P}[\widetilde{J}_j \geq k\delta].$$

Taking $D_i := \widetilde{J}_i - \widetilde{J}_{i-1}$ for $i \geq 1$ and observing that $(D_i)_{i\geq 1}$ is an i.d. sequence we have that

$$\mathbb{P}[\nu^Y > k, S_k \geq \delta k, S_{a_{l-1}} < \delta k, S_{a_l} \geq k\delta] \leq c_1^{l-1}\mathbb{P}[\widetilde{J}_l \geq k\delta] \leq c_1^{l-1} l \mathbb{P}\big[D_1 \geq \frac{k\delta}{l}\big].$$

Claim 2.3.3. *There exists constant $c_6 > 0$ such that for every $x > 0$ we have that*

$$\mathbb{P}[D_1 \geq x] \leq \frac{c_6}{\sqrt{x}}.$$

Proof of Claim 2.3.3. As in [6] take $\mu := \mathbb{E}[\widetilde{R}_0]$ and $\mathcal{J}_m := \inf\{t \geq 0 : B(t) = m\}$. Then,

$$\begin{aligned}
\mathbb{P}[D_1 \geq x] &= \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \mathbb{P}[D_1 \geq x | \widetilde{R}_0 = k, \widetilde{R}_1 = j]\mathbb{P}[\widetilde{R}_0 = k, \widetilde{R}_1 = j] \\
&= \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \mathbb{P}[D_1 \geq x | \widetilde{R}_0 = k, \widetilde{R}_1 = j]\mathbb{P}[\widetilde{R}_0 = k]\mathbb{P}[\widetilde{R}_1 = j] \\
&= \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \mathbb{P}[\mathcal{J}_{k+j} \geq x]\mathbb{P}[\widetilde{R}_0 = k]\mathbb{P}[\widetilde{R}_1 = j] \\
&= \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \int_x^{\infty} \frac{j+k}{\sqrt{2\pi y^3}} e^{-\frac{j+k}{2y}} dy \mathbb{P}[\widetilde{R}_0 = k]\mathbb{P}[\widetilde{R}_1 = j] \\
&\leq 2\mu^2 \int_x^{\infty} \frac{1}{\sqrt{2\pi y^3}} e^{-\frac{1}{2y}} dy \\
&\leq \frac{c_6}{\sqrt{x}}.
\end{aligned}$$

For some suitable constant c_6. ☐

Using Claim 2.3.3 we have some constant c_7 such that

$$\mathbb{P}[\nu^Y > k, S_k \geq \delta k, S_{a_{l-1}} < \delta k, S_{a_l} \geq k\delta] \leq \frac{c_7 c_1^l l^{\frac{3}{2}}}{\sqrt{k}},$$

then

$$\mathbb{P}[\nu^Y > k, S_k > \delta k] \leq \sum_{l=1}^{k} \frac{c_7 c_1^l l^{\frac{3}{2}}}{\sqrt{k}} \leq \frac{c_7}{\sqrt{k}} \sum_{l=1}^{\infty} c_1^l l^{\frac{3}{2}} = \frac{c_8}{\sqrt{k}}. \tag{2.3.13}$$

Then we get that

$$\mathbb{P}[\nu^Y > k] \leq \frac{c_5 + c_8}{\sqrt{k}}.$$

☐

CHAPTER

THREE

PROOF OF THE MAIN RESULT

In this chapter we will prove Theorem 1.2.1. To do this, it is enough to verify conditions I, B, E and T of Theorem 1.1.2, for the sequence $(\bar{\mathcal{X}}_n)_{n\geq 1}$. The main difference to prove I, compared with others works, is how we deal with crossing : the coupling constructed in Proposition 3.1.2. Condition B does not follow immediately from Proposition 2.3.1 as usually because, for any $t_0 \in \mathbb{R}$, we have to take care with the paths that start at time before t_0 and could cross $[a, b]$ at time t_0. A version of the renewal result given previously will be needed here. Getting this control of the paths starting before some time t_0, condition E follows using some ideas and results developed in [12]. Finally the proof of condition T is an adaptation of the proof given in [12].

3.1 Condition I

In this section we will prove condition I of Theorem 1.1.2. First we need to obtain the constants γ and σ such that π^u_n, as defined in (1.2.3), converges in distribution to a Brownian motion. Corollary 2.2.1 makes the proof of the convergence of π^u_n analogous to the one made in [13], without nothing new to highlight. Finally to get condition I, we will follow the ideas used in [6]. The principal difference in this part is the way we constructed the coupling.

Proposition 3.1.1. *There exist positive constants γ and σ such that for any $u \in \mathbb{Z}^2$ the rescaled path π^u_n, as defined in (2), converges in distribution, to a Brownian motion.*

Proof. Without lost of generality we can assume that $u = (0, 0)$. To make easier the notation we will omit u from the notation, i.e. we will write X_n instead of $X_n(u)$, $\pi(t)$ instead of $\pi^u(t)$ and so on. Taking $T_0 := 0, \tau_0 := 0$ and $(T_n)_{n\geq 1}$, $(\tau_n)_{n\geq 1}$ as defined in Corollary 2.2.1 for one point. Let $\widetilde{\pi}$ be the linear interpolation of the values of $(\pi(t))_{t\geq 0}$ on the renewals times,

$$\widetilde{\pi}(t) := \pi(T_n) + \frac{t - T_n}{T_{n+1} - T_n}\Big[\pi(T_{n+1}) - \pi(T_n)\Big], \text{ for } T_n \leq t \leq T_{n+1}.$$

Let us define the following random variables

$$Y_i := \pi(T_i) - \pi(T_{i-1}) \text{ for } i \geq 1.$$

$$S_0 := 0, S_n := \sum_{i=1}^{n} Y_i \text{ for } n \geq 1.$$

Let $\sigma^2 = \mathbb{V}ar(Y_1)$, then by Donsker's invariance principle we have that $(\widehat{\pi}_n(t))_{t\geq 0}$, defined as

$$\widehat{\pi}_n(0) := 0,\, \widehat{\pi}_n(t) := \frac{1}{n\sigma}\Big[(n^2 t - \lfloor n^2 t\rfloor)Y_{\lfloor n^2 t\rfloor+1} + S_{\lfloor n^2 t\rfloor}\Big] \text{ for } t > 0\,,$$

converges in distribution as $n \to \infty$ to a Standard Brownian motion $(B(t))_{t\geq 0}$.
Put

$$A(t) := j + \frac{t - T_j}{T_{j+1} - T_j} \text{ for } T_j \leq t < T_{j+1}, \text{ and } N(t) := \sup\{n \geq 1; T_n \leq t\} \text{ for } t > 0.$$

Note that $N(t) \leq A(t) \leq N(t)+1$ for all $t > 0$. Since $T_n = \sum_{j=1}^n (T_j - T_{j-1})$, by Corollary 2.2.1, $(N(t))_{t\geq 0}$ is a renewal process. By the Renewal Theorem $\frac{N(t)}{t} \to \frac{1}{\mathbb{E}[T_1]}$ as $t \to \infty$ almost surely, hence for $\gamma = \mathbb{E}[T_1]$ we have that $\frac{A(n^2\gamma t)}{n^2} \to t$ almost surely. For $n \geq 1$ let us rescale $\widetilde{\pi}$ as

$$\widetilde{\pi}_n(t) := \frac{\widetilde{\pi}(\gamma n^2 t)}{n\sigma} \text{ for } t \geq 0.$$

Note that

$$\widetilde{\pi}_n(t) = \widehat{\pi}_n\Big(\frac{A(n^2\gamma t)}{n^2}\Big) \text{ for } t \geq 0.$$

We have that $(\widetilde{\pi}_n(t))_{t\geq 0}$ converges in distribution to a $(B(t))_{t\geq 0}$. To prove the convergence of $(\pi_n(t))_{t\geq 0}$ to $(B(t))_{t\geq 0}$ it is enough to show that for any $\epsilon > 0$ and $s > 0$, $\mathbb{P}\big[\sup_{0\leq t\leq s} |\pi_n(t) - \widetilde{\pi}_n(t)| > \epsilon\big] \to 0$ as $n \to \infty$. Note that

$$\mathbb{P}\Big[\sup_{0\leq t\leq s} |\pi_n(t) - \widetilde{\pi}_n(t)| > \epsilon\Big] = \mathbb{P}\Big[\sup_{0\leq t\leq sn^2\gamma} |\pi(t) - \widetilde{\pi}(t)| > \epsilon n\sigma\Big].$$

Moreover

$$\Big\{\sup_{0\leq t\leq sn^2\gamma} |\pi(t) - \widetilde{\pi}(t)| > \epsilon n\sigma\Big\} \subseteq \bigcup_{j=0}^{N(sn^2\gamma)} \Big\{\sup_{T_j\leq t\leq T_{j+1}} |\pi(t) - \widetilde{\pi}(t)| > \epsilon n\sigma\Big\}$$

and $N(sn^2\gamma) \leq \lfloor sn^2\gamma\rfloor$ so,

$$\Big\{\sup_{0\leq t\leq sn^2\gamma} |\pi(t) - \widetilde{\pi}(t)| > \epsilon n\sigma\Big\} \subseteq \bigcup_{j=0}^{\lfloor sn^2\gamma\rfloor} \Big\{\sup_{T_j\leq t\leq T_{j+1}} |\pi(t) - \widetilde{\pi}(t)| > \epsilon n\sigma\Big\}.$$

Since π and $\widetilde{\pi}$ coincide at the renewal times and their increments are stationary then

$$\mathbb{P}\Big[\sup_{0\leq t\leq s} |\pi_n(t) - \widetilde{\pi}_n(t)| > \epsilon\Big] \leq (\lfloor sn^2\gamma\rfloor + 1)\mathbb{P}\Big[\sup_{0\leq t\leq T_1} \{|\pi(t) - \widetilde{\pi}(t)|\} > \epsilon n\sigma\Big].$$

Note that $\pi(T_1) = \widetilde{\pi}(T_1)$ and $\sup_{0\leq t\leq T_1} |\pi(t) - \pi(T_1)| \leq Z$, $\sup_{0\leq t\leq T_1} |\widetilde{\pi}(t) - \pi(T_1)| \leq Z$, where Z is defined in Proposition 2.2.1. Then

$$\begin{aligned}\mathbb{P}\Big[\sup_{0\leq t\leq s} |\pi_n(t) - \widetilde{\pi}_n(t)| > \epsilon\Big] &\leq (\lfloor sn^2\gamma\rfloor + 1)\mathbb{P}[2Z > \epsilon n\sigma] \\ &\leq \frac{2^3(\lfloor sn^2\gamma\rfloor + 1)\mathbb{E}[Z^3]}{\epsilon^3\sigma^3 n^3} \to 0 \text{ as } n \to \infty.\end{aligned}$$

□

Proposition 3.1.2. *Let $\mathcal{X}_n$ be defined as in* (1.2.3) *where the constants γ and σ are taken as in Proposition 3.1.1. Then for any $y_1, \dots, y_m \in \mathbb{R}^2$ there exist paths $\theta_n^{y_1}, \dots, \theta_n^{y_m}$ in $\mathcal{X}_n$, such that $(\theta_n^{y_1}, \dots, \theta_n^{y_m})$ converges in distribution as $n \to \infty$ to coalescing Brownian motions starting in $y_1, \dots, y_m$.*

To prove Proposition 3.1.2 we will use a coupling argument. To build the coupling, we will need Proposition 3.1.3 below, which is a version of Proposition 2.2.1 that will be presented without proof, since its proof follows the same lines as those of Proposition 2.2.1.

Proposition 3.1.3. *Let $\{U_v^1; v \in \mathbb{Z}^2\}$, $\{U_v^2; v \in \mathbb{Z}^2\}$, $\{W_v^1; v \in \mathbb{Z}^2\}$ and $\{W_v^2; v \in \mathbb{Z}^2\}$ be i.i.d. families independent of each other such that the U_v^j, $j = 1, 2$, are Uniform random variables in $[0, 1]$ and of W_v^j, $j = 1, 2$, are identically distributed positives random variables on $\mathbb{N}$ with finite support. Consider the GRDF systems*

$$\mathcal{X}^1 := \{\pi^{1,v}, v \in \mathbb{Z}^2\} \text{ and } \mathcal{X}^2 := \{\pi^{2,v}; v \in \mathbb{Z}^2\}$$

built respectively using the random variables $\{\{U_v^1; v \in \mathbb{Z}^2\}, \{W_v^1; v \in \mathbb{Z}^2\}\}$ and $\{\{U_v^2; v \in \mathbb{Z}^2\}, \{W_v^2; v \in \mathbb{Z}^2\}\}$. Then for points $u_1^1, \dots, u_{m_1}^1$ and $u_1^2, \dots, u_{m_2}^2$ in $\mathbb{Z}^2$ at the same time level, i.e. with equal second component, there exist random variables T, Z and $\tau(u_i^j)$ for $j = 1, 2$, $1 \le i \le m_j$, such that $T \le Z$ and

(i) $\Delta^j_{\tau(u_i^j)}(u_i) = \emptyset$ and $X^1_{\tau(u_1^1)}(u_1^1)(2) = X^j_{\tau(u_i^j)}(u_i^j)(2)$ for $j = 1, 2$ and $i = 1, \dots, m_j$. Where for $j = 1, 2$ and $v \in \mathbb{Z}^2$ the sequence $\{X_k^j(v)\}_{k \ge 0}$ is as defined in (1.2) using the r.v. $\{U_v^j; v \in \mathbb{Z}^2\}$, $\{W_v^j; v \in \mathbb{Z}^2\}$ and $\{\Delta_k^j(v)\}_{k \ge 0}$ as defined in (2.2.1) for the sequence $\{X_k^j(v)\}_{k \ge 0}$.

(ii) Taking $T := X^1_{\tau(u_1^1)}(u_1^1)(2)$ we have that its distribution depends on $m_1 + m_2$ but not on $u_1^j, \dots, u_{m_j}^j$, $j = 1, 2$. For all $k \ge 1$ we get $\mathbb{E}[T^k] < \infty$. Note that $\pi^{j,u_i^j}(T) = X^j_{\tau(u_i^j)}(u_i^j)(1)$ for $j = 1, 2$, $1 \le i \le m_j$.

(iii) For all $j = 1, 2$ and $1 \le i \le m_j$ $i = 1, \dots, m$ we have that $\sup_{0 \le t \le T} |\pi^{j,u_i^j}(t) - u_i^j(1)| \le Z$ and its distribution depends on $m_1 + m_2$ but not on $u_1^j, \dots, u_{m_j}^j$ for $j = 1, 2$. Also for all $k \ge 1$ we get $\mathbb{E}[Z^k] < \infty$.

Proof of the Proposition 3.1.2. Here we use a non-straightforward adaptation of the idea applied in [6] to prove condition I for the Drainage Network model. We will prove that for any $m \in \mathbb{N}$,

$$(\pi_n^{(0,0)}, \pi_n^{(n\sigma,0)}, \dots, \pi_n^{(mn\sigma,0)})$$

converges in distribution to a vector of coalescing Brownian motions starting in $(0, 0)$, $\dots$, $(m, 0)$ denoted here by $(B^{(0,0)}, \dots, B^{(m,0)})$. The general case, where the paths do not start necessarily at the same time, could be proved using the same technique, so we will omit it. To simplify the notation we will write $\pi^k := \pi^{(k,0)}$, $k \in \mathbb{Z}$, and $B^x := B^{(x,0)}$ for $x \in \mathbb{R}$. Here for the rescaled paths we use the notation:

$$\pi_n^k = \frac{\pi^{k\lfloor n\sigma \rfloor}(tn^2\gamma)}{n\sigma}.$$

It is enough to fix an arbitrary $M > 0$, suppose that $(B^0, \dots, B^m)$ and $(\pi_n^0, \pi_n^1, \dots, \pi_n^m)$ are restricted to time interval $[0, M]$ and prove the convergence, i.e.,

$$\lim_{n\to\infty} (\pi_n^0(t), \pi_n^1(t), \dots, \pi_n^m(t))_{0\le t\le M} \stackrel{d}{=} \left(B^0(t), \dots, B^m(t)\right)_{0\le t\le M} \tag{3.1.1}$$

By Proposition 3.1.1 we have that

$$\lim_{n\to\infty} (\pi^0(t))_{0\le t\le M} \stackrel{d}{=} \left(B^0(t)\right)_{0\le t\le M}.$$

Now we are going to make an induction in m. Let us suppose that

$$\lim_{n\to\infty} (\pi_n^0(t), \pi_n^1(t), \dots, \pi_n^{(m-1)}(t))_{0\le t\le M} \stackrel{d}{=} \left(B^0(t), \dots, B^{(m-1)}(t)\right)_{0\le t\le M}.$$

Now the proof of (3.1.1) from the induction hypothesis will be based on coupling techniques. We will build a path $\bar{\pi}_n^m$ which is independent of $(\pi_n^0, \pi_n^1, \dots, \pi_n^{(m-1)})$ until coalescence with one of them, has the same distribution of π_n^m and besides that, in a proper way, π_n^m and $\bar{\pi}_n^m$ will be close to each other.

We start constructing paths $\widetilde{\pi}^0$, ... , $\widetilde{\pi}^{(m-1)\lfloor n\sigma\rfloor}$ and $\widehat{\pi}^{m\lfloor n\sigma\rfloor}$ that coincide with $\widetilde{\pi}^0, \dots,$ $\widetilde{\pi}^{(m-1)\lfloor n\sigma\rfloor}$, $\widehat{\pi}^{m\lfloor n\sigma\rfloor}$ until one of these paths moves a distance $n^{\frac{3}{4}}$ from its last position on the renewal times, we suggest the reader to see Figure 7 although some definitions are still missing. The construction follows by induction:

Step 1: Let $\{\widetilde{U}_v; v \in \mathbb{Z}^2\}$ and $\{\widehat{U}_v; v \in \mathbb{Z}^2\}$ be i.i.d. families of Uniform r.v. in $[0, 1]$; $\{\widetilde{W}_v; v \in \mathbb{Z}^2\}$ and $\{\widehat{W}_v; v \in \mathbb{Z}^2\}$ be i.i.d families of r.v. with the same distribution of $W_{(0,0)}$; independent of each other and of $\{U_v; v \in \mathbb{Z}^2\}$ and $\{W_v; v \in \mathbb{Z}^2\}$. Using them let us define the r.v. $\{\widetilde{U}_v^1; v \in \mathbb{Z}^2\}$, $\{\widehat{U}_v^1; v \in \mathbb{Z}^2\}$, $\{\widetilde{W}_v^1; v \in \mathbb{Z}^2\}$ and $\{\widehat{W}_v^1; v \in \mathbb{Z}^2\}$ as follows:

$$\widehat{U}_v^1 := \begin{cases} U_v; \text{ if } |v(1) - mn\sigma| \le n^{\frac{3}{4}} \text{ and } 0 < v(2) \le n^{\frac{3}{4}}; \\ \widehat{U}_v; \text{ otherwise,} \end{cases}$$

$$\widehat{W}_v^1 := \begin{cases} W_v; \text{ if } |v(1) - mn\sigma| \le n^{\frac{3}{4}} \text{ and } 0 < v(2) \le n^{\frac{3}{4}}; \\ \widehat{W}_v; \text{ otherwise,} \end{cases}$$

$$\widetilde{W}_v^1 := \begin{cases} W_v; \text{ if } v(1) \le (m-1)n\sigma + n^{\frac{3}{4}} \text{ and } 0 < v(2) \le n^{\frac{3}{4}}; \\ \widetilde{W}_v; \text{ otherwise,} \end{cases}$$

and

$$\widetilde{U}_v^1 := \begin{cases} U_v; \text{ if } v(1) \le (m-1)n\sigma + n^{\frac{3}{4}} \text{ and } 0 < v(2) \le n^{\frac{3}{4}}; \\ \widetilde{U}_v; \text{ otherwise.} \end{cases}$$

Use the families $\{\widehat{U}_v^1; v \in \mathbb{Z}^2\}$, $\{\widehat{W}_v^1; v \in \mathbb{Z}^2\}$ to construct a path $\widehat{\pi}^{m\lfloor n\sigma\rfloor}$ of the GRDF (not rescaled) starting in $m\lfloor n\sigma\rfloor$ at time zero. Also use $\{\widetilde{U}_v^1; v \in \mathbb{Z}^2\}$, $\{\widetilde{W}_v^1; v \in \mathbb{Z}^2\}$ to construct paths $\{\widetilde{\pi}^0, \dots, \widetilde{\pi}^{(m-1)\lfloor n\sigma\rfloor}\}$ of the GRDF (not rescaled) starting respectively in $0, \lfloor n\sigma\rfloor, \dots, (m-1)\lfloor n\sigma\rfloor$ at time zero. Let T_1 and Z_1 be the random variables associated to $\{\widetilde{\pi}^0, \dots, \widetilde{\pi}^{(m-1)\lfloor n\sigma\rfloor}, \widehat{\pi}^{m\lfloor n\sigma\rfloor}\}$ by Proposition 3.1.3. Note that on the event $\{Z_1 \le n^{\frac{3}{4}}\}$ the vector paths $(\widetilde{\pi}^0, \dots, \widetilde{\pi}^{(m-1)\lfloor n\sigma\rfloor}, \widehat{\pi}^{m\lfloor n\sigma\rfloor})$ coincide with $(\pi^0, \dots, \pi^{m\lfloor n\sigma\rfloor})$ up to time $T_1 \le Z_1$. This ends Step 1.

Step 2: See that nothing above T_1 is known, so we can use other random variables to define the paths after this time. So from time T_1, we define new independent iid families $\{\widetilde{U}_v^2; v \in \mathbb{Z}^2\}$, $\{\widehat{U}_v^2; v \in \mathbb{Z}^2\}$, $\{\widetilde{W}_v^2; v \in \mathbb{Z}^2\}$ and $\{\widehat{W}_v^2; v \in \mathbb{Z}^2\}$ as follows:

$$\widehat{U}_v^2 := \begin{cases} U_v; \text{ if } |v(1) - \widehat{\pi}^{1,m\lfloor n\sigma \rfloor}(T_1)| \leq n^{\frac{3}{4}} \text{ and } T_1 < v(2) \leq T_1 + n^{\frac{3}{4}}; \\ \widehat{U}_v; \text{ otherwise,} \end{cases}$$

$$\widehat{W}_v^2 := \begin{cases} W_v; \text{ if } |v(1) - \widehat{\pi}^{1,m\lfloor n\sigma \rfloor}(T_1)| \leq n^{\frac{3}{4}} \text{ and } T_1 < v(2) \leq T_1 + n^{\frac{3}{4}}; \\ \widehat{W}_v; \text{ otherwise,} \end{cases}$$

$$\widetilde{W}_v^2 := \begin{cases} W_v; \text{ if } v(1) \leq \max_{0 \leq j \leq m-1} \widetilde{\pi}^{1,j\lfloor n\sigma \rfloor}(T_1) + n^{\frac{3}{4}} \text{ and } T_1 < v(2) \leq T_1 + n^{\frac{3}{4}}; \\ \widetilde{W}_v; \text{ otherwise,} \end{cases}$$

and

$$\widetilde{U}_v^2 := \begin{cases} U_v; \text{ if } v(1) \leq \max_{0 \leq j \leq m-1} \widetilde{\pi}^{1,j\lfloor n\sigma \rfloor}(T_1) + n^{\frac{3}{4}} \text{ and } T_1 < v(2) \leq T_1 + n^{\frac{3}{4}}; \\ \widetilde{U}_v; \text{ otherwise.} \end{cases}$$

Now consider $\widehat{\pi}^{2,m\lfloor n\sigma \rfloor}$ as the GRDF path starting in $\widehat{\pi}^{m\lfloor n\sigma \rfloor}(T_1)$ at time T_1 using the environment $\{\widehat{U}_v^2; v \in \mathbb{Z}^2\}$, $\{\widehat{W}_v^2; v \in \mathbb{Z}^2\}$, and $\widetilde{\pi}^{2,0}, \widetilde{\pi}^{2,\lfloor n\sigma \rfloor}, \ldots, \widetilde{\pi}^{(m-1)\lfloor n\sigma \rfloor}$ starting respectively in $\widetilde{\pi}^{0}(T_1), \widetilde{\pi}^{\lfloor n\sigma \rfloor}(T_1), \ldots, \widetilde{\pi}^{(m-1)\lfloor n\sigma \rfloor}(T_1)$ and using the environment $\{\widetilde{U}_v^2, v \in \mathbb{Z}^2\}$,$\{\widetilde{W}_v^2; v \in \mathbb{Z}^2\}$. Again we have random variables T_2 and Z_2 for these paths as in Proposition 3.1.3 and on the event $\{\max(Z_1, Z_2) \leq n^{\frac{3}{4}}\}$ the vector $(\widetilde{\pi}^0, \ldots, \widetilde{\pi}^{(m-1)\lfloor n\sigma \rfloor}, \widehat{\pi}^{m\lfloor n\sigma \rfloor})$ coincide with $(\pi^0, \ldots, \pi^{m\lfloor n\sigma \rfloor})$ up to time $T_2 \leq Z_1 + Z_2$. Redefine, if necessary, $(\widetilde{\pi}^0, \widetilde{\pi}^{\lfloor n\sigma \rfloor}, \ldots, \widetilde{\pi}^{(m-1)\lfloor n\sigma \rfloor})$ as $(\widetilde{\pi}^{2,0}, \widetilde{\pi}^{2,\lfloor n\sigma \rfloor}, \ldots, \widetilde{\pi}^{(m-1)\lfloor n\sigma \rfloor})$ on time interval $T_1 < t \leq T_2$. This ends Step 2.

We continue step by step replicating recursively Step k from Step k-1. We get $(T_k)_{k \geq 1}$, $(Z_k)_{k \geq 1}$ and $\{\widetilde{\pi}^{k,0}, \widetilde{\pi}^{k,\lfloor n\sigma \rfloor}, \ldots, \widetilde{\pi}^{k,(m-1)\lfloor n\sigma \rfloor}, \widehat{\pi}^{k,m\lfloor n\sigma \rfloor}\}$ for $k \geq 1$ such that on the event $\{\max(Z_1, ..., Z_k) \leq n^{\frac{3}{4}}\}$ the vector $(\widetilde{\pi}^0, \ldots, \widetilde{\pi}^{(m-1)\lfloor n\sigma \rfloor}, \widehat{\pi}^{m\lfloor n\sigma \rfloor})$ coincide with $(\pi^0, \ldots, \pi^{m\lfloor n\sigma \rfloor})$ up to time $T_k \leq \sum_{j=1}^{k} Z_j$.

Now let us define a version $\bar{\pi}^{m\lfloor n\sigma \rfloor}$ of $\widehat{\pi}^{m\lfloor n\sigma \rfloor}$ such that it is independent of $(\widetilde{\pi}^0, \ldots, \widetilde{\pi}^{(m-1)\lfloor n\sigma \rfloor})$ and coincide with $\widehat{\pi}^{m\lfloor n\sigma \rfloor}$ until this last path gets to distance $2n^{3/4}$ of $(\widetilde{\pi}^0, \ldots, \widetilde{\pi}^{(m-1)\lfloor n\sigma \rfloor})$. Consider the following stopping time

$$\nu := \inf\Big\{k \geq 1; \max_{0 \leq j \leq m-1} |\widehat{\pi}^{m\lfloor n\sigma \rfloor}(T_k) - \widetilde{\pi}^{j\lfloor n\sigma \rfloor}(T_k)| \leq 2n^{\frac{3}{4}}\Big\}.$$

Define $\bar{\pi}^{m\lfloor n\sigma \rfloor}(t) = \widehat{\pi}^{m\lfloor n\sigma \rfloor}(t)$ for $0 \leq t \leq T_\nu$, see Figure 7. From time T_ν we have that $\bar{\pi}^{m\lfloor n\sigma \rfloor}(t)$ evolves only through the environment $(\{\widehat{U}_v; v \in \mathbb{Z}^2\}, \{\widehat{W}_v; v \in \mathbb{Z}^2\})$ as the path starting in $\widehat{\pi}^{m\lfloor n\sigma \rfloor}(T_\nu)$ at time T_ν before coalescence with some $\widetilde{\pi}^0, \ldots, \widetilde{\pi}^{(m-1)\lfloor n\sigma \rfloor}$. Let

$$\widetilde{\pi}_n^j(t) := \frac{\widetilde{\pi}^{j\lfloor n\sigma \rfloor}(tn^2\gamma)}{n\sigma} \text{ for } j = 0, \ldots, m-1$$

and

$$\widehat{\pi}_n^m(t) := \frac{\widehat{\pi}^{m\lfloor n\sigma \rfloor}(t)}{n\sigma}, \bar{\pi}_n^m(t) := \frac{\bar{\pi}^{m\lfloor n\sigma \rfloor}(t)}{n\sigma}$$

the rescaled versions of the constructed paths.

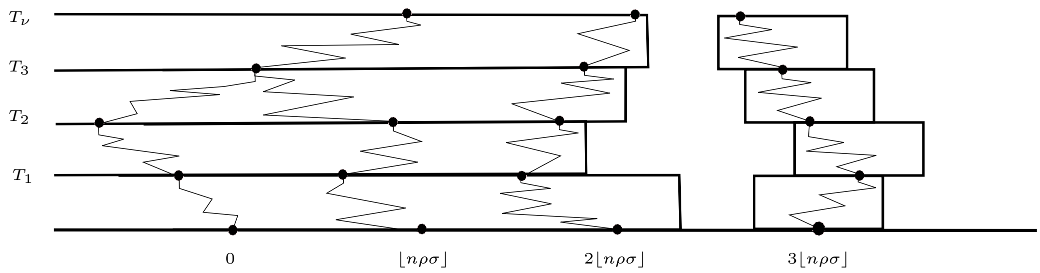

Figure 3.1: Here m=4 and we consider the GRDF paths π^0, $\pi^{\lfloor n\sigma \rfloor}$, $\pi^{2\lfloor n\sigma \rfloor}$. In the picture $\pi^{3\lfloor n\sigma \rfloor}$ remains at distance $n^{\frac{3}{4}}$ of its position on the previous renewal time. Moreover none of π^0, $\pi^{\lfloor n\sigma \rfloor}$, $\pi^{2\lfloor n\sigma \rfloor}$ go beyond $n^{\frac{3}{4}}$ to the right of their rightmost position at the previous renewal time. In such scenario, $\nu = 4$ and before time T_4 we have that $(\pi^0, \pi^{\lfloor n\sigma \rfloor}, \pi^{2\lfloor n\sigma \rfloor}, \pi^{3\lfloor n\sigma \rfloor})$ coincide with $(\widetilde{\pi}^0, \widetilde{\pi}^{\lfloor n\sigma \rfloor}, \widetilde{\pi}^{2\lfloor n\sigma \rfloor}, \bar{\pi}^{3\lfloor n\sigma \rfloor})$.

Remark 3.1.1. *We point out that as a direct consequence of the definitions the following properties are satisfied:*

(i) Before coalescence, the path $\bar{\pi}^m_n$ *is independent of* $\widetilde{\pi}^0_n, \dots, \widetilde{\pi}^{(m-1)}_n$.

(ii) For $s \leq M$*, on the event*

$$\mathcal{A}_{n,s} := \{T_\nu > n^2\gamma s\},$$

we have that $\widehat{\pi}^m_n(t) = \bar{\pi}^m_n(t)$ *for every* $0 \leq t \leq s$.

(iii) From the induction hypothesis, item (i) and Proposition 3.1.1 we get

$$\lim_{n\to\infty} \big(\widetilde{\pi}^0_n, \dots, \widetilde{\pi}^{(m-1)}_n, \bar{\pi}^m_n\big) \overset{d}{=} \big(B^0, \dots, B^m\big).$$

(iv) On the event

$$\mathcal{B}_{n,M} := \cap_{k=1}^{\lfloor Mn^2\gamma \rfloor + 1} \{Z_k \leq n^{\frac{3}{4}}\}$$

the vector of paths $(\widetilde{\pi}^0, \dots, \widetilde{\pi}^{(m-1)}, \widehat{\pi}^m)$ *coincide with* $(\pi^0, \dots, \pi^m)$ *up to a time greater than* $Mn^2\gamma$.

(v) Also on $\mathcal{B}_{n,M}$*, if* $|\widehat{\pi}^{m\lfloor n\sigma \rfloor}(t) - \widetilde{\pi}^{j\lfloor n\sigma \rfloor}(t)| \leq 2n^{\frac{3}{4}}$ *for some* $0 \leq j \leq m-1$ *and* $t > 0$ *then either there exists some* k *such that* $T_k < t$, $|\widehat{\pi}^{m\lfloor n\sigma \rfloor}(T_k) - \widetilde{\pi}^{j\lfloor n\sigma \rfloor}(T_k)| \leq 2n^{\frac{3}{4}}$ *and* $\nu \leq k$ *or* $\widehat{\pi}^{m\lfloor n\sigma \rfloor}$ *and* $\widetilde{\pi}^{j\lfloor n\sigma \rfloor}$ *cannot coalesce or cross each other before* $\min\{T_k : T_k > t\}$.

Claim 3.1.1. *For the event* $\mathcal{B}_{n,M}$ *as in Remark 3.1.1 we have that* $\lim_{n\to\infty} \mathbb{P}\big[\mathcal{B}^c_{n,M}\big] = 0$.

Proof. Note that

$$\mathbb{P}\big[\mathcal{B}^c_{n,M}\big] \leq (Mn^2\gamma + 1)\mathbb{P}\big[Z_1 > n^{\frac{3}{4}}\big] \leq \frac{(Mn^2\gamma + 1)\mathbb{E}[Z_1^4]}{n^3}$$

which goes to zero as n goes to infinity. □

Now let $H : C^{k+1}[0, M] \to \mathbb{R}$ be an uniformly continuous function. We need to prove that

$$\lim_{n\to\infty} \mathbb{E}\big[H\big(\pi^0_n, \dots, \pi^m_n\big)\big] = \mathbb{E}\big[H\big(B^0, \dots, B^m\big)\big].$$

By Remark 3.1.1 and Claim 3.1.1 we have that

$$\mathbb{E}\Big[\big|H\big(\pi_n^0,\dots,\pi_n^m\big)-H\big(\widetilde{\pi}_n^0,\dots,\widetilde{\pi}_n^{(m-1)},\widehat{\pi}_n^m\big)\big|\Big]\le 2||H||_\infty\mathbb{P}\big[\mathcal{B}_{n,M}^c\big]\to 0 \text{ as n goes to infinity.}$$

And also by Remark 3.1.1 and the induction hypothesis we have that

$$\mathbb{E}\Big[H\big(\widetilde{\pi}_n^0,\dots,\widetilde{\pi}_n^{(m-1)},\bar{\pi}_n^m\big)\Big]\to\mathbb{E}\Big[H\big(B^0,\dots,B^m\big)\Big].$$

By triangular inequality

$$\begin{aligned}&\Big|\mathbb{E}\big[H\big(\pi_n^0,\dots,\pi_n^m\big)\big]-\mathbb{E}\big[H\big(B^0,\dots,B^m\big)\big]\Big|\\&\le\mathbb{E}\Big[\big|H\big(\pi_n^0,\dots,\pi_n^m\big)-H\big(\widetilde{\pi}_n^0,\dots,\widetilde{\pi}_n^{(m-1)},\widehat{\pi}_n^m\big)\big|\Big]\\&+\mathbb{E}\Big[\big|H\big(\widetilde{\pi}_n^0,\dots,\widetilde{\pi}_n^{(m-1)},\widehat{\pi}_n^m\big)-H\big(\widetilde{\pi}_n^0,\dots,\widetilde{\pi}_n^{(m-1)},\bar{\pi}_n^m\big)\big|\Big]\\&+\Big|\mathbb{E}\Big[H\big(\widetilde{\pi}_n^0,\dots,\widetilde{\pi}_n^{(m-1)},\bar{\pi}_n^m\big)-H\big(B^0,\dots,B^m\big)\Big]\Big|,\end{aligned}$$

then it is enough to prove that

$$\lim_{n\to\infty}\mathbb{E}\Big[\big|H\big(\widetilde{\pi}_n^0,\dots,\widetilde{\pi}_n^{(m-1)},\widehat{\pi}_n^m\big)-H\big(\widetilde{\pi}_n^0,\dots,\widetilde{\pi}_n^{(m-1)},\bar{\pi}_n^m\big)\big|\Big]=0\,.$$

Note that

$$\begin{aligned}&\mathbb{E}\Big[\big|H\big(\widetilde{\pi}_n^0,\dots,\widetilde{\pi}_n^{(m-1)},\widehat{\pi}_n^m\big)-H\big(\widetilde{\pi}_n^0,\dots,\widetilde{\pi}_n^{(m-1)},\bar{\pi}_n^m\big)\big|\Big]\\&=\mathbb{E}\Big[\big|H\big(\widetilde{\pi}_n^0,\dots,\widetilde{\pi}_n^{(m-1)},\widehat{\pi}_n^m\big)-H\big(\widetilde{\pi}_n^0,\dots,\widetilde{\pi}_n^{(m-1)},\bar{\pi}_n^m\big)\big|\mathbb{1}_{\mathcal{A}_{n,M}^c}\Big]\\&\le\mathbb{E}\Big[\big|H\big(\pi_n^0,\dots,\pi_n^{(m-1)},\pi_n^m\big)-H\big(\pi_n^0,\dots,\pi_n^{(m-1)},\bar{\pi}_n^m\big)\big|\mathbb{1}_{\mathcal{A}_{n,M}^c}\mathbb{1}_{\mathcal{B}_{n,M}}\Big]+2||H||_\infty\mathbb{P}\big[\mathcal{B}_{n,M}^c\big].\end{aligned}$$

Again, by Claim 3.1.1 we just have to prove that

$$\lim_{n\to\infty}\mathbb{E}\Big[\big|H\big(\pi_n^0,\dots,\pi_n^{(m-1)},\pi_n^m\big)-H\big(\pi_n^0,\dots,\pi_n^{(m-1)},\bar{\pi}_n^m\big)\big|\mathbb{1}_{\mathcal{A}_{n,M}^c}\mathbb{1}_{\mathcal{B}_{n,M}}\Big]=0\,.$$

Before we are able to obtain the above convergence, we need to define some stopping times. For $j=\{0,\dots,m-1\}$ consider

$$\nu_j:=\inf\{k\ge 1:|\pi^{j\lfloor n\sigma\rfloor}(T_k)-\pi^{m\lfloor n\sigma\rfloor}(T_k)|\le 2n^{\frac{3}{4}}\}\,,$$

where the definition is based on (v) in Remark 3.1.1 from where we see that on $\mathcal{B}_{n,M}$ we only need to consider approximation between paths on the renewal times. Then

$$\begin{aligned}&\mathbb{E}\Big[\big|H\big(\pi_n^0,\dots,\pi_n^{(m-1)},\pi_n^m\big)-H\big(\pi_n^0,\dots,\pi_n^{(m-1)},\bar{\pi}_n^m\big)\big|\mathbb{1}_{\mathcal{A}_{n,M}^c}\mathbb{1}_{\mathcal{B}_{n,M}}\Big]\\&\le\sum_{j=0}^{m-1}\mathbb{E}\Big[\big|H\big(\pi_n^0,\dots,\pi_n^{(m-1)},\pi_n^m\big)-H\big(\pi_n^0,\dots,\pi_n^{(m-1)},\bar{\pi}_n^m\big)\big|\mathbb{1}_{\mathcal{A}_{n,M}^c}\mathbb{1}_{\mathcal{B}_{n,M}}\mathbb{1}_{\{\nu=\nu_j\}}\Big].\end{aligned}$$

Given $\epsilon>0$, since H is uniformly continuous, there exists $\delta_\epsilon>0$ such that: if $\|f-g\|_\infty\le\delta_\epsilon$ for $f,g\in C^{k=1}[0,M]$, then $\big|H(f)-H(g)\big|\le\epsilon$. So, if

$$\sup_{0\le t\le M}|\pi_n^m(t)-\bar{\pi}_n^m(t)|\le\delta_\epsilon$$

we get
$$\Big|H\big(\pi_n^0,\dots,\pi_n^{(m-1)},\pi_n^m\big)-H\big(\pi_n^0,\dots,\pi_n^{(m-1)},\bar\pi_n^m\big)\Big|\le\epsilon.$$
To simplify the notation let us denote $D_{n,j}:=\mathcal{A}_{n,M}^c\cap\mathcal{B}_{n,M}\cap\{\nu=\nu_j\}$. For $j=0,\dots,m-1$ we have that
$$\begin{aligned}&\mathbb{E}\Big[\big|H\big(\pi_n^0,\dots,\pi_n^{(m-1)},\pi_n^m\big)-H\big(\pi_n^0,\dots,\pi_n^{(m-1)},\bar\pi_n^m\big)\big|\mathbb{1}_{D_{n,j}}\Big]\\&\le\epsilon+2\|H\|_\infty\mathbb{P}\Big[D_{n,j}\cap\{\sup_{0\le t\le M}|\pi_n^m(t)-\bar\pi_n^m(t)|>\delta_\epsilon\}\Big]\\&=\epsilon+2\|H\|_\infty\mathbb{P}\Big[D_{n,j}\cap\{\sup_{0\le t\le Mn^2\gamma}|\pi^{m\lfloor n\sigma\rfloor}(t)-\bar\pi^{m\lfloor n\sigma\rfloor}(t)|>n\sigma\delta_\epsilon\}\Big].\end{aligned}$$
For $j=0,\dots,m-1$ let us define
$$\tau^j:=\inf\{t>0:\pi^{j\lfloor n\sigma\rfloor}(s)=\pi^{m\lfloor n\sigma\rfloor}(s),\ \forall s\ge t\}$$
and
$$\bar\tau^j:=\inf\{t>0:\pi^{j\lfloor n\sigma\rfloor}(s)=\bar\pi^{m\lfloor n\sigma\rfloor}(s),\ \forall s\ge t\}.$$
Fix some $\beta\in(\frac{3}{2},2)$. Then for $j=0,\dots,m-1$ and n large enough
$$\mathbb{P}\Big[D_{n,j}\cap\{\sup_{0\le t\le Mn^2\gamma}|\pi^{m\lfloor n\sigma\rfloor}(t)-\bar\pi^{m\lfloor n\sigma\rfloor}(t)|>n\sigma\delta_\epsilon\}\Big]$$
is bounded above by
$$\begin{aligned}&\mathbb{P}\Big[D_{n,j}\cap\{\sup_{0\le t\le Mn^2\gamma}|\pi^{m\lfloor n\sigma\rfloor}(t)-\bar\pi^{m\lfloor n\sigma\rfloor}(t)|>n\sigma\delta_\epsilon\}\cap\{\tau^j,\bar\tau^j\in[T_\nu,T_\nu+n^\beta\gamma]\}\Big]\\&+\mathbb{P}\Big[D_{n,j}\cap\{\tau^j>T_\nu+n^\beta\gamma]\}\Big]+\mathbb{P}\Big[D_{n,j}\cap\{\bar\tau^j>T_\nu+n^\beta\gamma]\}\Big].\end{aligned}\tag{3.1.2}$$
The first probability in (3.1.2) is equal to
$$\mathbb{P}\Big[D_{n,j}\cap\{\sup_{T_{\nu_j}\le t\le Mn^2\gamma\wedge(T_{\nu_j}+n^\beta\gamma)}|\pi_n^m(t)-\bar\pi_n^m(t)|>n\sigma\delta_\epsilon\}\cap\{\tau^j,\bar\tau^j\in[T_{\nu_j},T_{\nu_j}+n^\beta\gamma]\}\Big]$$
which is bounded above by
$$\begin{aligned}&\mathbb{P}\Big[D_{n,j}\cap\{\sup_{T_{\nu_j}\le t\le Mn^2\gamma\wedge(T_{\nu_j}+n^\beta\gamma)}|\pi_n^m(t)-\pi_n^m(T_{\nu_j})|>\frac{n\sigma\delta_\epsilon}{2}\}\cap\{\tau^j,\bar\tau^j\in[T_{\nu_j},T_{\nu_j}+n^\beta\gamma]\}\Big]\\&+\\&\mathbb{P}\Big[D_{n,j}\cap\{\sup_{T_{\nu_j}\le t\le Mn^2\gamma\wedge(T_{\nu_j}+n^\beta\gamma)}|\bar\pi_n^m(t)-\bar\pi_n^m(T_{\nu_j})|>\frac{n\sigma\delta_\epsilon}{2}\}\cap\{\tau^j,\bar\tau^j\in[T_{\nu_j},T_{\nu_j}+n^\beta\gamma]\}\Big].\end{aligned}$$
The first term is bounded by
$$\mathbb{P}\Big[\sup_{0\le t\le n^\beta\gamma}|\pi^{m\lfloor n\sigma\rfloor}(t)-\pi^{m\lfloor n\sigma\rfloor}(0)|>\frac{n\sigma\delta_\epsilon}{2}\Big]\tag{3.1.3}$$
and the second one by
$$\mathbb{P}\Big[\sup_{0\le t\le n^\beta\gamma}|\bar\pi^{m\lfloor n\sigma\rfloor}(t)-\bar\pi^{m\lfloor n\sigma\rfloor}(0)|>\frac{n\sigma\delta_\epsilon}{2}\Big].\tag{3.1.4}$$

For the inequalities we have used the Markov property on the renewal times. Both terms, (3.1.3) and (3.1.4) are bounded above by

$$\mathbb{P}\Big[\sup_{0\le t\le n^{\beta}\gamma}\frac{|\pi^0(t)-\pi^0(0)|}{\sigma n^{\frac{\beta}{2}}}>\frac{n^{1-\frac{\beta}{2}}\delta_\epsilon}{2}\Big],$$

which, by the choice of $\beta<2$ and the invariance principle proved in Proposition 3.1.1, converges to zero as $n\to\infty$. Thus the first probability in (3.1.2) converges to zero as $n\to\infty$.

Now it remains to deal with the second and third terms in (3.1.2). Since

$$|\pi^{j\lfloor n\sigma\rfloor}(T_{v_j})-\pi^{m\lfloor n\sigma\rfloor}(T_{v_j})|\le 2n^{\frac{3}{4}}$$

on $D_{n,j}$, then by Corollary 2.3.1 there is some constant C such that

$$\mathbb{P}\Big[D_{n,j}\cap\{\tau^j>T_{\nu_j}+n^{\beta}\gamma\}\Big]\le\frac{2Cn^{\frac{3}{4}}}{n^{\frac{\beta}{2}}}$$

which converges to zero as $n\to\infty$ by the choice of $\beta>3/2$. Even though $\pi^{j\lfloor n\sigma\rfloor}$ and $\bar{\pi}^{j\lfloor n\sigma\rfloor}$ are independent from time T_{ν_j} until coalescence, we can prove the result stated in Corollary 2.3.1 for these paths, following the same lines of the proof of that corollary. Thus we get a constant $\bar{C}$ such that

$$\mathbb{P}\Big[D_{n,j}\cap\{\bar{\tau}^j>T_{\nu_j}+n^{\beta}\gamma\}\Big]\le\frac{2\bar{C}n^{\frac{3}{4}}}{n^{\frac{\beta}{2}}},$$

which as before converges to zero as $n\to\infty$.

Hence

$$\lim_{n\to\infty}\mathbb{P}\Big[\mathcal{A}^c_{n,M}\cap\mathcal{B}_{n,M}\cap\{\nu=\nu_j\}\cap\{\sup_{0\le t\le Mn^2\gamma}|\pi^m_n(t)-\bar{\pi}^m_n(t)|>n\sigma\delta_\epsilon\}\Big]=0$$

which finishes the proof. □

3.2 Condition B

We prove condition B of Theorem 1.1.2 at the end of this section. Before we prove it we need to introduce some definitions and establish some preliminary results.

We are going to need another result about renewal times. Here we need to define the renewal times for a finite collection of paths in $\mathcal{X}^{t^-}$ such that all we know about them is that they cross an interval $[a,b]$ at time t. Therefore we are interested in $\{\pi\in\mathcal{X}^{t^-}:\pi(t)\in[a,b]\}$ which is almost surely finite by Lemma 2.1.2 since it is the set of paths in $\mathcal{X}^{t^-}$ whose projection at time t is in $\mathcal{X}^{t^-}(t)\cap[a,b]$.

Lemma 3.2.1. *Fix $a<b$ and consider the collection of paths $\Gamma=\{\pi\in\mathcal{X}^{t^-}:\pi(t)\in[a,b]\}$. For any $\pi,\pi^0\in\Gamma$ there exist random variables T_{π,π^0} and Z_{π,π^0} such that*

(i) $t<T_{\pi,\pi^0}$ and $T_{\pi,\pi^0}-t\le Z_{\pi,\pi^0}$.

(ii) T_{π,π^0} is a common renewal time for π_1 and π_2.

(iii) $\max_{\pi \in \{\pi, \pi^0\}} \sup_{t \le s \le T_{\pi,\pi^0}} |\pi(s) - \pi(t)| \le Z_{\pi,\pi^0}$.

(iv) *For all* $k \in \mathbb{N}$ *we have that* $\mathbb{E}\big[(Z_{\pi,\pi^0})^k\big] < \infty$.

(v) *For any others* $\widetilde{\pi}, \widetilde{\pi}^0 \in \Gamma$ *we have that* $T_{\widetilde{\pi},\widetilde{\pi}^0}$ *and* T_{π,π^0} *are i.d given that* $\big|\mathcal{X}^{t^-}(t) \cap [a,b]\big|$. *The same happens for* $Z_{\widetilde{\pi},\widetilde{\pi}^0}$ *and* Z_{π,π^0}.

Proof. Define $\xi_0 := \max\big\{H\big((\lfloor a \rfloor, t)\big), H\big((\lfloor b \rfloor + 1, t)\big)\big\}$, where H as defined in (2.2.2). Note that all random variables associated to the points above ξ_0, are independents to the condition that $\pi(t)$ and $\pi^0(t)$ belong to $[a,b]$. So, run each paths up to the first time that they need to see above ξ_0 to decide where to jump. After that repeat the procedure made in Proposition 2.2.1 to get the random variables T_{π,π^0} and Z_{π,π^0}. $\square$

We need one more result before we prove condition B of Theorem 1.1.2.

Lemma 3.2.2. *There exists a constant* $C_1 > 0$ *such that*

$$\mathbb{P}\big[|\eta_{\mathcal{X}}(0,k,0,m)| > 1\big] \le \frac{C_1 m}{\sqrt{k}},$$

for every $m \ge 1$.

Proof. Note that

$$\mathbb{P}\big[|\eta_{\mathcal{X}}(0,k,0,m)| > 1\big] \le \sum_{i=1}^{m} \mathbb{P}\big[|\eta_{\mathcal{X}(0,k,i-1,i)}| > 1\big] = m\mathbb{P}\big[|\eta_{\mathcal{X}(0,k,0,1)}| > 1\big],$$

and

$$\begin{aligned} &\mathbb{P}\big[|\eta_{\mathcal{X}}(0,k,0,1)| > 1\big] \\ &= \sum_{j=2}^{\infty} \mathbb{P}\big[|\eta_{\mathcal{X}}(0,k,0,1)| > 1 \big| |\mathcal{X}^{0^-}(0) \cap [0,1]| = j\big]\mathbb{P}\big[|\mathcal{X}^{0^-}(0) \cap [0,1]| = j\big] \end{aligned} \tag{3.2.1}$$

Given $|\mathcal{X}^{0^-}(0) \cap [0,1]| = j$, let $\pi_1, \dots, \pi_j$ be the paths in $\mathcal{X}^{0^-}$ such that $\pi_i(0) \in [0,1]$ for $i = 1, \dots, j$ and define

$$\nu_{i,i+1} := \inf\{n \ge 1 : \pi_i(t) = \pi_{i+1}(t), \text{ for all } t \ge n\}.$$

See that

$$\mathbb{P}\big[|\eta_{\mathcal{X}}(0,k,0,1)| > 1 \big| |\mathcal{X}^{0^-}(0) \cap [0,1]| = j\big] \le \sum_{i=1}^{j-1} \mathbb{P}\big[\nu_{i,i+1} > k \big| |\mathcal{X}^{0^-}(0) \cap [0,1]| = j\big]. \tag{3.2.2}$$

For $i = 1, \ldots, j-1$ let $T_{i,i+1}$ and $Z_{i,i+1}$ be random variables as in Lemma 3.2.1 for the paths π_i and π_{i+1}. Then we have that

$$
\begin{aligned}
&\mathbb{P}\Big[\nu_{i,i+1} > k \Big| \big|\mathcal{X}^{0^-}(0) \cap [0,1]\big| = j\Big] \\
&= \mathbb{P}\Big[\nu_{i,i+1} > k, T_{i,i+1} > \frac{k}{2} \Big| \big|\mathcal{X}^{0^-}(0) \cap [0,1]\big| = j\Big] + \qquad (3.2.3) \\
&+ \mathbb{P}\Big[\nu_{i,i+1} > k, T_{i,i+1} \leq \frac{k}{2} \Big| \big|\mathcal{X}^{0^-}(0) \cap [0,1]\big| = j\Big] \\
&\leq \mathbb{P}\Big[T_{i,i+1} > \frac{k}{2} \Big| \big|\mathcal{X}^{0^-}(0) \cap [0,1]\big| = j\Big] + \mathbb{P}\Big[\nu_{i,i+1} > k, T_{i,i+1} \leq \frac{k}{2} \Big| \big|\mathcal{X}^{0^-}(0) \cap [0,1]\big| = j\Big] \\
&\leq \frac{2\mathbb{E}\Big[T_{i,i+1} \Big| \big|\mathcal{X}^{0^-}(0) \cap [0,1]\big| = j\Big]}{k} + \mathbb{P}\Big[\nu_{i,i+1} > k, T_{i,i+1} \leq \frac{k}{2} \Big| \big|\mathcal{X}^{0^-}(0) \cap [0,1]\big| = j\Big] \qquad (3.2.4) \\
&\leq \frac{2\mathbb{E}\Big[T_{1,2} \Big| \big|\mathcal{X}^{0^-}(0) \cap [0,1]\big| = j\Big]}{k} + \mathbb{P}\Big[\nu_{i,i+1} > k, T_{i,i+1} \leq \frac{k}{2} \Big| \big|\mathcal{X}^{0^-}(0) \cap [0,1]\big| = j\Big]. \qquad (3.2.5)
\end{aligned}
$$

Now define $\nu^T_{i,i+1} := \inf\{t \geq T_{i,i+1}; \pi_i(s) = \pi_{i+1}(s) \text{ for all } s \geq t\}$. Then we have that

$$
\begin{aligned}
&\mathbb{P}\Big[\nu_{i,i+1} > k, T_{i,i+1} \leq \frac{k}{2} \Big| \big|\mathcal{X}^{0^-}(0) \cap [0,1]\big| = j\Big] \\
&= \sum_{l=1}^{\infty} \mathbb{P}\Big[\nu_{i,i+1} > k, T_{i,i+1} \leq \frac{k}{2}, \big|\pi_i(T_{i,i+1}) - \pi_{i+1}(T_{i,i+1})\big| = l \Big| \big|\mathcal{X}^{0^-}(0) \cap [0,1]\big| = j\Big] \\
&\leq \sum_{l=1}^{\infty} \mathbb{P}\Big[\nu^T_{i,i+1} > \frac{k}{2}, \big|\pi_i(T_{i,i+1}) - \pi_{i+1}(T_{i,i+1})\big| = l \Big| \big|\mathcal{X}^{0^-}(0) \cap [0,1]\big| = j\Big] \\
&= \sum_{l=1}^{\infty} \mathbb{P}\Big[\nu^T_{i,i+1} > \frac{k}{2} \Big| \big|\pi_i(T_{i,i+1}) - \pi_{i+1}(T_{i,i+1})\big| = l\Big] * \\
&* \mathbb{P}\Big[\big|\pi_i(T_{i,i+1}) - \pi_{i+1}(T_{i,i+1})\big| = l \Big| \big|\mathcal{X}^{0^-}(0) \cap [0,1]\big| = j\Big]. \qquad (3.2.6)
\end{aligned}
$$

By Collorary 2.3.1 we have that

$$
\mathbb{P}\Big[\nu^T_{i,i+1} > \frac{k}{2} \Big| \big|\pi_i(T_{i,i+1}) - \pi_{i+1}(T_{i,i+1})\big| = l\Big] \leq \frac{2lC}{\sqrt{k}}. \qquad (3.2.7)
$$

Replacing (3.2.7) in (3.2.6) we get

$$
\begin{aligned}
&\mathbb{P}\Big[\nu_{i,i+1} > k, T_{i,i+1} \leq \frac{k}{2} \Big| \big|\mathcal{X}^{0^-}(0) \cap [0,1]\big| = j\Big] \\
&\leq \sum_{l \geq 1} \frac{2lC}{\sqrt{k}} \mathbb{P}\Big[\big|\pi_i(T_{i,i+1}) - \pi_{i+1}(T_{i,i+1})\big| = l \Big| \big|\mathcal{X}^{0^-}(0) \cap [0,1]\big| = j\Big] \\
&= \frac{2C}{\sqrt{k}} \mathbb{E}\Big[\big|\pi_i(T_{i,i+1}) - \pi_{i+1}(T_{i,i+1})\big| \Big| \big|\mathcal{X}^{0^-}(0) \cap [0,1]\big| = j\Big]. \qquad (3.2.8)
\end{aligned}
$$

(3.2.9)

Since

$$\begin{aligned}
&\big|\pi_i(T_{i,i+1}) - \pi_{i+1}(T_{i,i+1})\big| \\
&\le \big|\pi_i(T_{i,i+1}) - \pi_i(0)\big| + \big|\pi_i(0) - \pi_{i+1}(0)\big| + \big|\pi_{i+1}(0) - \pi_{i+1}(T_{i,i+1})\big| \\
&\le 2Z_{i,i+1} + 1\,,
\end{aligned}$$

we have that (3.2.8) is bounded above by

$$\frac{2C}{\sqrt{k}}\mathbb{E}\Big[2Z_{i,i+1}+1\Big|\big|\mathcal{X}^{0^-}(0)\cap[0,1]\big|=j\Big] = \frac{2C}{\sqrt{k}}\mathbb{E}\Big[2Z_{1,2}+1\Big|\big|\mathcal{X}^{0^-}(0)\cap[0,1]\big|=j\Big]. \quad (3.2.10)$$

Now replacing (3.2.10) in (3.2.3) we obtain,

$$\begin{aligned}
&\mathbb{P}\Big[\nu_{i,i+1}>k\Big|\big|\mathcal{X}^{0^-}(0)\cap[0,1]\big|=j\Big] \\
&\le \frac{2\mathbb{E}\Big[Z_{1,2}\Big|\big|\mathcal{X}^{0^-}(0)\cap[0,1]\big|=j\Big]}{k} + \frac{2C}{\sqrt{k}}\mathbb{E}\Big[2Z_{1,2}+1\Big|\big|\mathcal{X}^{0^-}(0)\cap[0,1]\big|=j\Big] \\
&\le \frac{2(1+C)\mathbb{E}\Big[2Z_{1,2}+1\Big|\big|\mathcal{X}^{0^-}(0)\cap[0,1]\big|=j\Big]}{\sqrt{k}}. \qquad (3.2.11)
\end{aligned}$$

Hence by (3.2.2) and (3.2.11),

$$\begin{aligned}
&\mathbb{P}\Big[\big|\eta_{\mathcal{X}}(0,k,0,1)\big|>1\Big|\big|\mathcal{X}^{0^-}(0)\cap[0,1]\big|=j\Big] \\
&\qquad \le \frac{2(1+C)}{\sqrt{k}}\, j\,\mathbb{E}\Big[2Z_{1,2}+1\Big|\big|\mathcal{X}^{0^-}(0)\cap[0,1]\big|=j\Big]. \qquad (3.2.12)
\end{aligned}$$

Replacing (3.2.12) in (3.2.1) we get that $\mathbb{P}[|\eta_{\mathcal{X}}(0,k,0,1)|>1]$ is dominated by

$$\frac{2(1+C)}{\sqrt{k}}\sum_{j=2}^{\infty} j\mathbb{E}\big[2Z_{1,2}+1\big||\mathcal{X}^{0^-}(0)\cap[0,1]|=j\big]\mathbb{P}\big[|\mathcal{X}^{0^-}(0)\cap[0,1]|=j\big] \qquad (3.2.13)$$

Note that

$$\begin{aligned}
&\sum_{j=2}^{\infty} j\mathbb{E}\big[2Z_{1,2}+1\big||\mathcal{X}^{0^-}(0)\cap[0,1]|=j\big]\mathbb{P}\big[|\mathcal{X}^{0^-}(0)\cap[0,1]|=j\big] \\
&\le \Big(\sum_{j=2}^{\infty} j^2\mathbb{P}\big[\big||\mathcal{X}^{0^-}(0)\cap[0,1]|=j\big]\Big)^{\frac{1}{2}} * \\
&* \Big(\sum_{j=2}^{\infty}\mathbb{E}\big[2Z_{1,2}+1\big||\mathcal{X}^{0^-}(0)\cap[0,1]|=j\big]^2\mathbb{P}\big[|\mathcal{X}^{0^-}(0)\cap[0,1]|=j\big]\Big)^{\frac{1}{2}} \\
&\le \Big(\sum_{j=2}^{\infty} j^2\mathbb{P}\big[\big||\mathcal{X}^{0^-}(0)\cap[0,1]|=j\big]\Big)^{\frac{1}{2}} * \\
&* \Big(\sum_{j=2}^{\infty}\mathbb{E}\big[(2Z_{1,2}+1)^2\big||\mathcal{X}^{0^-}(0)\cap[0,1]|=j\big]\mathbb{P}\big[|\mathcal{X}^{0^-}(0)\cap[0,1]|\big]\Big)^{\frac{1}{2}} \\
&= \mathbb{E}\Big[\big|\mathcal{X}^{0^-}(0)\cap[0,1]\big|^2\Big]^{\frac{1}{2}}\mathbb{E}\big[(2Z_{1,2}+1)^2\big]^{\frac{1}{2}}. \qquad (3.2.14)
\end{aligned}$$

Take $C_1 := 2(1+C)\mathbb{E}\big[|\mathcal{X}^{0^-}(0)\cap[0,1]|^2\big]^{\frac{1}{2}}\mathbb{E}\big[(2Z_{1,2}+1)^2\big]^{\frac{1}{2}}$ which is finite by Lemma 2.1.2 and Lemma 3.2.1. Replacing (3.2.14) in (3.2.13) we have that

$$\mathbb{P}[|\eta_{\mathcal{X}}(0,k,0,1)|>1]\leq\frac{C_1}{\sqrt{k}},$$

which finishes the proof. □

Proof of the condition B of the Theorem 1.1.2. Fix $\epsilon>0$ and take M_ϵ as in the statement of Lemma 2.1.3, from that result we get that

$$\sup_{t_0,a\in\mathbb{R}}\mathbb{P}\big[|\eta_{\mathcal{X}_n}(t_0,t,a-\epsilon,a+\epsilon)|>1\big]$$

is bounded above by

$$\mathbb{P}\big[|\eta_{\mathcal{X}}(0,n^2\gamma t,n\sigma(a-\epsilon)-M_\epsilon,n\sigma(a+\epsilon)+M_\epsilon)|>1\big]+\epsilon\,.$$

Then by Lemma 3.2.2

$$\sup_{t>\beta}\sup_{t_0,a\in\mathbb{R}}\mathbb{P}\big[|\eta_{\mathcal{X}_n}(t_0,t,a-\epsilon,a+\epsilon)|>1\big]\leq\frac{C_1}{n\sqrt{\gamma\beta}}2(n\sigma\epsilon+M_\epsilon)+\epsilon.$$

Hence

$$\limsup_{n\to\infty}\sup_{t>\beta}\sup_{t_0,a\in\mathbb{R}}\mathbb{P}\big[|\eta_{\mathcal{X}_n}(t_0,t,a-\epsilon,a+\epsilon)|>1\big]\leq\Big(\frac{2C_1\sigma}{\sqrt{\beta\gamma}}+1\Big)\epsilon\to0\text{ as }\epsilon\to0^+.$$

So we have condition B. □

3.3 Condition E

This section is dedicated to condition E in Theorem 1.1.2. We will follows the proof given in [12] to Coalescing Nonsimple Random Walks.

Let us start with a powerful tool for the analyzing properties of the Brownian web: is its dual. The dual Brownian web $\widehat{W}$ is a collection of paths which is distributed as the Brownian web running back in time. This random elemnt takes values in the following space: for each $\pi=(f,t_0)\in\Pi$ consider $\widehat{\pi}=(\widehat{f},-t_0)$ where $\widehat{f}(s):=f(-s)$ for $s\leq-t_0$. Denote by $\widehat{\Pi}$ the sets of all such backwards paths and consider a metric $\widehat{d}$ induced from (π,d). Let $\widehat{W}$ be the space of compact subsets of $(\widehat{\Pi},\widehat{d})$ with the Hausdorff metric $d_{\widehat{\mathcal{H}}}$ and Borel $\sigma-$álgebra $\mathcal{B}_{\widehat{\mathcal{H}}}$. For a $\widehat{K}\in\widehat{\mathcal{H}}$ we will denote by $-\widehat{K}$ the set $\{\pi\in\Pi:\widehat{\pi}\in\widehat{K}\}$. The next theorem, that prove the existence of $\widehat{W}$, was proved in [9]. You can also find it in [14].

Theorem 3.3.1. *There exists an $\mathcal{H}\times\widehat{\mathcal{H}}-$valued random variable $(\mathcal{W},\widehat{\mathcal{W}})$, called the double Brownian web, whose distribution is uniquely determined by the following properties:*

(a) $\mathcal{W}$ and $-\widehat{\mathcal{W}}$ are both distributed as the Brownian web.

(b) Almost surely, no path $\pi_z\in W$ crosses any path $\widehat{\pi}_{\widehat{z}}\in\widehat{\mathcal{W}}$ in sense that that, $z=(x,t)$ and $\widehat{z}=(\widehat{x},\widehat{t})$ with $\widehat{t}<t$, and $(\pi_z(s_1)-\widehat{\pi}_{\widehat{z}}(s_1))(\pi_z(s_2)-\widehat{\pi}_{\widehat{z}}(s_2))<0$ for some $t<s_1<s_2<\widehat{t}$.

Furthermore, for each $z \in \mathbb{R}^2, \widehat{\mathcal{W}}(z)$ a.s. consists of a single path $\widehat{\pi}_z$ which is unique path in $\widehat{\Pi}$ that does not cross any path in $\mathcal{W}$, and thus $\widehat{\mathcal{W}}$ is a.s. determined by $\mathcal{W}$ and vice versa.

Now let us define the following counting random variable. For $t_0 \in \mathbb{R}, t > 0, a < b \in \mathbb{R}$ consider

$$\widehat{\eta}(t, t_0; a, b) := \big\{y \in (a,b) \times \{t_0 + t\}; \text{are touched by paths which also touch some point in } \mathbb{R} \times \{t_0\}\big\}.$$

Remark 3.3.1. *Note that by the duality we have that $\widehat{\eta}(t, t_0; a, b) \overset{d}{=} \eta(t, t_0; a, b) - 1$. Then the condition (E) in Theorem 1.1.2 could be written as:*

$(\widehat{E})$ If Z_{t_0} is the subsequential limit of $\{\mathcal{Y}_n^{t_0^-}\}_{n\geq 1}$ for any t_0 in $\mathbb{R}$, then for all t, a, b in $\mathbb{R}$ with $t > 0$ and $a < b$ we get

$$\mathbb{E}\big[|\widehat{\eta}_{Z_{t_0}}(t_0, t, a, b)|\big] \leq \frac{b-a}{\sqrt{\pi t}}.$$

Let us see that the following Lemma 3.3.1 implies the condition E. Before enunciate this lemma we will need introduce some definitions. For a set of paths $Y \subset \Pi$ define

(i) $Y(t) = \{\pi(t) : \pi \in Y\}$

(ii) Y^{s^-} = the subset of paths in Y such that start before or at time s;

(iii) For $A \subset \mathbb{R}$ define $Y^{s^-,A} = \{\pi \in Y^{s-} : \pi(s) \in A\}$;

(iv) For $s \leq t$ and $A \subset \mathbb{R}$ define $Y^{s-}(t) = \{\pi(t) : \pi \in Y^{s-}\}$ and $Y^{s-,A}(t) = \{\pi(t) : \pi \in Y^{s-,A}\}$.

Lemma 3.3.1. *Let Z_{t_0} be any subsequential limit of $(\bar{\mathcal{X}}_n^{t_0^-})_{n\geq 1}$ and $\epsilon > 0$. Denote by $\mathcal{Z}_{t_0}^{(t_0+\epsilon)T}$ the set of paths in Z_{t_0} that start before time t truncated before time $t_0 + \epsilon$. Then $\mathcal{Z}_{t_0}^{(t_0+\epsilon)T}$ is distributed as coalescing Brownian motions starting from the random set $\mathcal{Z}_{t_0}(t_0 + \epsilon) \subset \mathbb{R}^2$.*

Proof of the condition $(\widehat{E})$ using Lemma 3.3.1. Since $\mathcal{Z}_{t_0}^{(t_0+\epsilon)T}$ is distributed as coalescing Brownian motions starting from the random set $\mathcal{Z}_{t_0}(t_0 + \epsilon) \subset \mathbb{R}^2$ we have for $0 < \epsilon < t$,

$$\begin{aligned}
\mathbb{E}\big[|\widehat{\eta}_{Z_{t_0}}(t_0, t, a, b)|\big] &= \mathbb{E}\big[|\widehat{\eta}_{\mathcal{Z}_{t_0}^{(t_0+\epsilon)T}}(t_0 + \epsilon, t - \epsilon, a, b)|\big] \\
&\leq \mathbb{E}\big[|\widehat{\eta}_{\mathcal{W}}(t_0 + \epsilon, t - \epsilon, a, b)|\big] \\
&= \frac{b-a}{\sqrt{\pi(t-\epsilon)}} \to \frac{b-a}{\sqrt{\pi t}} \text{ as } \epsilon \to 0^+.
\end{aligned}$$

To see the last equality, notice that, by the Monotone Convergence Theorem we get

$$\mathbb{E}\big[|\widehat{\eta}_{\mathcal{W}}(0,t,a,b)|\big] = \lim_{n\to\infty} \mathbb{E}\Big[\big|\{j \in \{1,\dots,(b-a)2^n\}\} : \mathcal{W}(t) \cap (a + \frac{j-1}{2^n}, a + \frac{j}{2^n}) \neq \emptyset\big|\Big]$$
$$= \lim_{n\to\infty} (b-a)2^n \mathbb{P}\big[\mathcal{W}(t) \cap (0, \frac{1}{2^n}) \neq \emptyset\big]. \qquad (3.3.1)$$

Using the duality we have that $\mathcal{W}(t) \cap (0, \frac{1}{2^n}) \neq \emptyset$ is equivalent that the paths $\widehat{\pi}^0$ and $\widehat{\pi}^{\frac{1}{2^n}}$ in $\widehat{W}$ starting in 0 and $\frac{1}{2^n}$ at time t, respectively, do not coalesce before time 0. Using the duality again we have that, if ν_n is the first time that π^0 and $\pi^{\frac{1}{2^n}}$, paths in W starting in 0 and $\frac{1}{2^n}$ at time 0, then

$$\mathbb{P}[\mathcal{W}(t) \cap (0, \frac{1}{2^n}) \neq \emptyset] = \mathbb{P}[\nu_n > t] = \mathbb{P}[\frac{1}{2^n} + \inf_{0\le s\le t}(B^2(s) - B^1(s)) > 0],$$

where $\big(B^1(s)\big)_{s\ge 0}$ and $\big(B^2(s)\big)_{s\ge 0}$ are two independent standard Brownian motions. Hence

$$\begin{aligned}\mathbb{P}\big[\mathcal{W}(t) \cap (0, \frac{1}{2^n}) \neq \emptyset\big] &= \mathbb{P}\big[\sup_{0\le s\le t}(B^1(s) - B^2(s)) < \frac{1}{2^n}\big] \\ &= \mathbb{P}\big[\sup_{0\le s\le t} B(s) < \frac{1}{2^n\sqrt{2}}\big] \\ &= \frac{\sqrt{2}}{\sqrt{t\pi}} \int_0^{\frac{1}{2^n\sqrt{2}}} e^{-\frac{x^2}{2t}} dx \qquad (3.3.2)\end{aligned}$$

Replacing (3.3.2) in (3.3.1) we get

$$\mathbb{E}\big[|\widehat{\eta}_{\mathcal{W}}(0,t,a,b)|\big] = (b-a) \lim_{n\to\infty} \frac{2^n\sqrt{2}}{\sqrt{t\pi}} \int_0^{\frac{1}{2^n\sqrt{2}}} e^{-\frac{x^2}{2t}} dx = \frac{b-a}{\sqrt{t\pi}}$$

□

To prove Lemma 3.3.1 we will need two more results: Lemma 3.3.2 and Lemma 3.3.3. Denote the space of compact subsets of $(\bar{R}^2, \rho)$ by $(\mathcal{P}, \rho_{\mathcal{P}})$, where $\rho_{\mathcal{P}}$ is the induced Hausdorff metric.

Lemma 3.3.2. *Let A_n and A be $(\bar{R}^2, \rho)$ by $(\mathcal{P}, \rho_{\mathcal{P}})$-valued random variables, where A is almost surely a locally finite set, A_n is almost surely a subset of $(\mathbb{Z}/n\sigma) \times (\mathbb{Z}/\gamma n^2)$ and A_n converges in distribution to A as n goes to infinity. Conditioned on A_n, let $\mathcal{X}_n^{A_n}$ be the GRDF starting from the point set A_n. Then as n goes to infinity, $\mathcal{X}_n^{A_n}$ converges in distribution to the process of coalescing Brownian motions starting from a random point set distributed as A.*

Lemma 3.3.3. *Let Z_{t_0} be any subsequential limit of $\{\bar{\mathcal{X}}_n^{t_0^-}\}$. For any $\epsilon > 0$, $Z_{t_0}(t_0 + \epsilon)$ is almost surely locally finite and*

$$\mathbb{E}\Big[\big|Z_{t_0}(t_0+\epsilon) \cap (a,b)\big|\Big] \le \frac{(b-a)C_4}{\sqrt{\epsilon}}.$$

Proof of Lemma 3.3.1 using lemmas 3.3.2 and 3.3.3. Let Z_{t_0} be the weak limit of a subsequence $\bar{\mathcal{X}}_{n_k}^{t_0^-}$. By Skorohood's representation theorem, we can assume that the convergence is almost sure. Then almost surely,

$$\rho_{\mathcal{P}}\big(\bar{\mathcal{X}}_{n_k}^{t_0^-}(t_0+\epsilon), Z_{t_0}(t_0+\epsilon)\big) \text{ and } d_{\mathcal{H}}(\bar{\mathcal{X}}_{n_k}^{t_0^-,(t_0+\epsilon)T}, Z_{t_0}^{(t_0+\epsilon)T})$$

goes to zero as n_k goes to infinity. Let $s(n_k) = \frac{\lceil (t_0+\epsilon)n^2\gamma \rceil}{n^2\gamma}$ be the first time on the rescaled lattice greater than or equal to $t_0+\epsilon$. Using the fact that the image of $Z_{t_0}^{(t_0+\epsilon)T}$ under (Φ, Ψ) is equicontinuous, it is not difficult to see that

$$\rho_{\mathcal{P}}\big(\bar{\mathcal{X}}_{n_k}^{t_0^-}(s(n_k)), Z_{t_0}(t_0+\epsilon)\big) \text{ and } d_{\mathcal{H}}(\bar{\mathcal{X}}_{n_k}^{t_0^-,(s(n_k))T}, Z_{t_0}^{(t_0+\epsilon)T})$$

almost surely goes to zero as n_k goes to infinity. On the other hand, $Z_{t_0}(t_0+\epsilon)$ is locally finite by Lemma 3.3.3 and $\bar{\mathcal{X}}_{n_k}^{t_0^-,(s(n_k))T}$ is distributed as GRDF on the rescaled lattice starting from $\bar{\mathcal{X}}_{n_k}^{t_0^-}(s(n_k))$. Therefore by Lemma 3.3.2 $\bar{\mathcal{X}}_{n_k}^{t_0^-,(s(n_k))T}$ converges weakly to $\mathcal{B}^{Z_{t_0}(t_0+\epsilon)}$ and $Z_{t_0}^{(t_0+\epsilon)T}$ is equally distributed with $\mathcal{B}^{Z_{t_0}(t_0+\epsilon)}$. □

Let us to prove Lemma 3.3.3.

Lemma 3.3.4. *There exists a constant C_2 such that*

$$\mathbb{E}\Big[\Big|\mathcal{X}^{0^-}(t) \cap [0,1)\Big|\Big] \le \frac{C_2}{\sqrt{t}}$$

for all $t > 0$.

Proof. Fix $M \in \mathbb{Z}_+$ arbitrarily and note that

$$\begin{aligned} M\mathbb{E}\Big[\Big|\mathcal{X}^{0^-}(t) \cap [0,1)\Big|\Big] &= \mathbb{E}\Big[\Big|\mathcal{X}^{0^-}(t) \cap [0,M)\Big|\Big] \\ &= \sum_{i\in\mathbb{Z}} \mathbb{E}\Big[\Big|\mathcal{X}^{0^-,[iM,(i+1)M)}(t) \cap [0,M)\Big|\Big] \\ &= \sum_{i\in\mathbb{Z}} \mathbb{E}\Big[\Big|\mathcal{X}^{0^-,[0,M)}(t) \cap [iM,(i+1)M)\Big)\Big|\Big] = \mathbb{E}\Big[\Big|\mathcal{X}^{0^-,[0,M)}(t)\Big|\Big], \end{aligned}$$

where the third equality above follows from the symmetry of the GRDF paths. Since $\mathcal{X}^{0^-,[0,M)}(0)$ has at least M points which are 0,1,2,...,M-1, then

$$\begin{aligned} &M\mathbb{E}\Big[\Big|\mathcal{X}^{0^-}(t) \cap [0,1)\Big|\Big] \\ &\qquad = \sum_{j=M}^{\infty} \mathbb{E}\Big[\big|\mathcal{X}^{0^-,[0,M)}(t)\big|\Big|\big|\mathcal{X}^{0^-,[0,M)}(0)\big| = j\Big]\mathbb{P}\Big[\big|\mathcal{X}^{0^-,[0,M)}(0)\big| = j\Big]. \end{aligned}$$

From here the proof is very close to that of Lemma 3.2.2, given $|\mathcal{X}^{0^-}(0) \cap [0,M)| = j$, $j \ge M$, let $\pi_1, \ldots, \pi_j$ be the paths in $\mathcal{X}^{0^-}$ such that $0 \le \pi_1(0) < \pi_2(0) < \ldots < \pi_j(0) < M$ for $i = 1, \ldots, j$ and define

$$\nu_{i,i+1} := \inf\{n \ge 1 : \pi_i(t) = \pi_{i+1}(t), \text{ for all } t \ge n\}.$$

Then

$$\mathbb{E}\Big[\big|\mathcal{X}^{0^-,[0,M)}(t)\big|\Big|\big|\mathcal{X}^{0^-,[0,M)}(0)\big| = j\Big] \leq \mathbb{E}\Big[1 + \sum_{i=1}^{j-1} 1_{\{\nu_{i,i+1}>t\}}\Big|\big|\mathcal{X}^{0^-,[0,M)}(0)\big| = j\Big].$$

Note that $|\pi_i(0) = \pi_{i+1}(0)| \leq 1$ because $0, 1, ..., M-1 \in \mathcal{X}^{0^-,[0,M)}(0)$, then by Lemma 3.2.1 and (3.2.11) we have that there exists a constant $C > 0$ and a integrable random variable Z, both not depending on M, such that

$$\begin{aligned} P\Big[\nu_{i,i+1} > t\Big|\big|\mathcal{X}^{0^-,[0,M)}(0)\big| = j\Big] &\leq \frac{2(1+C)}{\sqrt{t}}\mathbb{E}\Big[2Z + 1\big|\big|\mathcal{X}^{0^-,[0,M)}(0)\big| = j\Big] \\ &\leq \frac{2(1+C)}{\sqrt{t}}\mathbb{E}\Big[3Z\big|\big|\mathcal{X}^{0^-,[0,M)}(0)\big| = j\Big] \\ &= \frac{\widetilde{C}}{\sqrt{t}}\mathbb{E}\Big[Z\big|\big|\mathcal{X}^{0^-,[0,M)}(0)\big| = j\Big]. \end{aligned}$$

Hence

$$M\mathbb{E}\Big[\big|\mathcal{X}^{0^-}(t) \cap [0,1)\big|\Big] \leq 1 + \frac{\widetilde{C}}{\sqrt{t}}\sum_{j=M}^{\infty} j\mathbb{E}\Big[Z\big|\mathcal{X}^{0^-,[0,M)}(0)\big| = j\Big]\mathbb{P}\Big[\big|\mathcal{X}^{0^-,[0,M)}(0)\big| = j\Big]$$

which as in (3.2.14) can be shown to be bounded above by

$$\begin{aligned} &1 + \frac{\widetilde{C}}{\sqrt{t}}\Big(\mathbb{E}\Big[\big|\mathcal{X}^{0^-,[0,M)}(0)\big|^2\Big]\Big)^{\frac{1}{2}}\Big(\mathbb{E}[(Z)^2]\Big)^{\frac{1}{2}} \\ &\qquad \leq 1 + \frac{\widetilde{C}}{\sqrt{t}}\Big(M\mathbb{E}\Big[\sum_{i=1}^{M}\big|\mathcal{X}^{0^-,[i-1,i]}(0)\big|^2\Big]\Big)^{\frac{1}{2}}\Big(\mathbb{E}[(Z)^2]\Big)^{\frac{1}{2}} \\ &\qquad = 1 + \frac{\widetilde{C}}{\sqrt{t}}M\Big(\mathbb{E}\Big[\big|\mathcal{X}^{0^-,[0,1]}(0)\big|^2\Big]\Big)^{\frac{1}{2}}\Big(\mathbb{E}[(Z)^2]\Big)^{\frac{1}{2}}. \end{aligned}$$

Thus

$$\mathbb{E}\Big[\big|\mathcal{X}^{0^-}(t) \cap [0,1)\big|\Big] \leq \frac{1}{M} + \frac{C_2}{\sqrt{t}},$$

where $C_2 := \widetilde{C}\Big(\mathbb{E}[(Z)^2]\Big)^{\frac{1}{2}}\Big(\mathbb{E}\Big[\big|\mathcal{X}^{0^-,[0,1]}(0)\big|^2\Big]\Big)^{\frac{1}{2}}$ which is finite by Lemma 2.1.2. Since M is arbitrary we obtain the bound in the statement. □

Lemma 3.3.5. *There exists a constant C_3, independent of M, such that*

$$\mathbb{E}\Big[\big|\mathcal{X}_n^{0^-}(t) \cap [0,M)\big|\Big] \leq \frac{MC_3}{\sqrt{t}},$$

for every $n \geq 1$ and $M \geq 1$.

Proof. Using Lemma 3.3.4 we have that for all $n \geq 1$

$$\begin{aligned} \mathbb{E}\Big[\big|\mathcal{X}^{0^-}(n^2\gamma t) \cap [0, n\sigma M)\big|\Big] &= n\sigma M\mathbb{E}\Big[\big|\mathcal{X}^{0^-}(n^2\gamma t) \cap [0,1)\big|\Big] \\ &\leq \frac{n\sigma MC_2}{\sqrt{n^2\gamma t}} = \frac{\gamma^{-1/2}\sigma MC_2}{\sqrt{t}}. \end{aligned}$$

□

Proof of Lemma 3.3.3. Let Z_{t_0} a weak limit of a sequence $\{\mathcal{X}_{n_k}^{t_0-}\}_{k\geq 1}$. Now for a $\mathcal{H}$-valued random element define the random variable

$$\widehat{\eta}'_X(t_0, t; a, b) = \lim_{s\downarrow 0} X^{(t_0-s)-}(t_0+t) \cap (a, b).$$

Note that $\{K \in \mathcal{H} : \widehat{\eta}'_X(t_0, t; a, b) \geq k\}$ is an open set for all $k \geq \mathbb{N}$. For $0 < \alpha < \epsilon$ and $a < b$ by the weak convergence and Lemma 3.3.5 we have that

$$\begin{aligned}
\mathbb{E}\big[|Z_{t_0}(t_0+\epsilon) \cap (a,b)|\big] &\leq \mathbb{E}\big[|\widehat{\eta}'_{Z_{t_0}}(t_0+\alpha, \epsilon-\alpha, a, b)|\big] \\
&= \sum_{j=1}^{\infty} \mathbb{P}\big[|\widehat{\eta}'_{Z_{t_0}}(t_0+\alpha, \epsilon-\alpha, a, b)| \geq j\big] \\
&\leq \sum_{j=1}^{\infty} \liminf_{k\to\infty} \mathbb{P}\big[|\widehat{\eta}'_{\mathcal{X}_{n_k}^{t_0-}}(t_0+\alpha, \epsilon-\alpha, a, b)| \geq j\big] \\
&\leq \liminf_{k\to\infty} \mathbb{E}\big[|\widehat{\eta}'_{\mathcal{X}_{n_k}^{t_0-}}(t_0+\alpha, \epsilon-\alpha, a, b)|\big] \\
&\leq \liminf_{k\to\infty} \mathbb{E}\big[|\mathcal{X}_{n_k}^{(t_0+\alpha)-}(t_0+\epsilon) \cap (a,b)|\big] \\
&\leq \frac{(b-a)C_3}{\sqrt{\epsilon-\alpha}}
\end{aligned}$$

Letting $\alpha \to 0$ we get Lemma 3.3.3. □

Proof of Lemma 3.3.2 we can find in [12] to Coalescing Nonsimple Random Walks, which could be completely adapted to the GRDF model without any extra difficulty, hence we will omit it here.

3.4 Condition T

In this section we will prove the condition T in Theorem 1.1.2 which follows from Proposition 3.4.1. The idea behind the proof comes from [12]. Technical details related to the renewals times impose an extra difficult.

Recall the definitions from the statement of condition T in Section. By homogeneity of the GRDF all the estimates on $A_{\mathcal{X}_n}(x_0, t_0; \rho, t)$ are uniform on $(x_0, t_0) \in \mathbb{R}^2$. Here we only consider $(x_0, t_0) = (0, 0)$ leaving the verification for other choices of t_0 to the reader. The case $n\gamma t_0 \notin \mathbb{Z}$ demands an extra care, but can be dealt analogously as done the previous sections to deal with paths crossing some time level not necessarily on the rescaled space/time lattice. With this in mind, condition T is a consequence of the next result.

Proposition 3.4.1. *Denote by $A^+_{\mathcal{X}_n}(x_0, t_0; \rho, t)$ the event that $\mathcal{X}_n$ contains a path touching both $R(x_0, t_0; \rho, t)$ and the right boundary of the rectangle $R(x_0, t_0; 20\rho, 4t)$. Then*

$$\lim_{t\to 0^+} \frac{1}{t} \limsup_{n\to\infty} \mathbb{P}\Big[A^+_{\mathcal{X}_n}(0, 0; \rho, t)\Big] = 0.$$

Before we prove Proposition 3.4.1 we need some lemmas. The first lemma gives an uniform bound on the overshoot distribution for random walks in Z_-. It is a result enunciated (and proved) as Lemma 2.6 in [12]. The second lemma is about the overshoot on the renewal times for paths in the GRDF: it is a consequence of the first lemma.

Lemma 3.4.1. *Let $Y^x = (Y_n^x)_{n\geq 0}$ be a random walk with increment Y, starting from $x < 0$ at time 0. For $r \in \mathbb{R}$, consider τ^x_{r+} as the first time the walk is above r, i.e.*

$$\tau^x_{r+} := \inf\{n \geq 1 : \xi^x_n \geq r\},$$

and $Z = \xi^0(\tau^0_{1+})$. If $\mathbb{E}[|Y|^{l+2}] < \infty$ for some $l > 0$, then $(Y^x(\tau^x_{0+}))^l$ is uniformly integrable in $x \in \mathbb{Z}_-$.

Lemma 3.4.2. *For $x \in \mathbb{Z}_-$ let $(T_i^x)_{i\geq 0}$ be a sequence of renewal times of the paths $\pi^{(x,0)}$ such that there exist a sequence of i.i.d. random variables $(Z_i^x)_{i\geq 1}$ with the following property*

$$\left|\pi^x(T_i^x) - \pi^x(T_{i-1}^x)\right| \leq Z_i^x \text{ for all } i \geq 1,$$

and $\mathbb{E}\big[(Z_1^x)^{k+2}\big] < \infty$ for a fixed $k \geq 1$. Define Y_i^x, $i \geq 0$, as the first component of the path $\pi^{(x,0)}$ on the random time T_i^x. Also define $\nu_+^x := \inf\{n \geq 1 : Y_n^x \geq 1\}$. Then we have that

$$\sup_{x\in\mathbb{Z}_-} \mathbb{E}\big[(Y^x_{\nu^x_+})^k\big] < \infty.$$

A path in the GRDF is obtained from linear interpolation between open points in $\mathbb{Z}^2$, we say that these open points defining the path are the ones visited by the path. The next Lemma states that the probability of having paths that cross a box $R(0,0;n\rho\sigma, n^2t\gamma)$ but do not visit any point in $R(0,0;2n\rho\sigma, 2n^2t\gamma)$ goes to zero as $n \to \infty$.

Lemma 3.4.3. *Let $D(n\rho\sigma, n^2t\gamma)$ be the event that paths in $\mathcal{X}$ cross $R(0,0;n\rho\sigma, n^2t\gamma)$ without visit any point in $R(0,0;2n\rho\sigma, 2n^2t\gamma)$, then*

$$\lim_{n\to\infty} \mathbb{P}\Big[D(n\rho\sigma, n^2t\gamma)\Big] = 0.$$

Proof. Recall from (2.2.2) the definition of the random variable

$$H(v) := \inf\left\{n \geq 1; \sum_{j=1}^{n} \mathbb{1}_{\{(v(1),v(2)+j) \text{ is open }\}} = K\right\},$$

where $v \in \mathbb{Z}^2$ and $K \in \mathbb{N}$ is such that $\mathbb{P}[W_v \leq K] = 1$. Put $H = H((0,0))$ and note that H has negative binomial distribution of parameters p and K, thus it has finite absolute moments of any order. If some path in $D(n\rho\sigma, n^2t\gamma)$ comes from $\mathbb{Z} \times \mathbb{Z}_-$ before crossing $R(0,0;2n\rho\sigma, 2n^2t\gamma)$ then $H(v) > n\rho\sigma$ for some $v \in \{-n\rho\sigma, \dots, n\rho\sigma\} \times \{0\}$, and

$$\begin{aligned}
&\mathbb{P}\big[H(v) > n\rho\sigma \text{ for some } v \in \{-n\rho\sigma, \dots, n\rho\sigma\} \times \{0\}\big] \\
&\leq 2n\rho\sigma\mathbb{P}[H > n\rho\sigma] \\
&\leq 2n\rho\frac{\mathbb{E}[H^2]}{(n\rho\sigma)^2} \to 0 \text{ as } n \text{ goes to infinity.}
\end{aligned}$$

The others paths in $D(n\rho\sigma, n^2t\gamma)$ come from points in with first component bigger than $2n\rho\sigma$ or smaller than $-2n\rho\sigma$ and second component between in $\{0, \dots, n^2t\gamma\}$. If from some $v = (v(1), v(2))$ with $v(1) > 2n\rho\sigma$ and $v(2) \in \{0, \dots, n^2t\gamma\}$ the path cross $R(0{,}0;2n\rho\sigma,\ 2n^2t\gamma)$ then $H(v) > v(1)$. In case that $v(1) < -2n\rho\sigma$ we have that $H(v) > -v(1)$. Then the probability that of one of these paths cross $R(0,0;2n\rho\sigma,2n^2t\gamma)$ is bounded by

$$\begin{aligned}
&2\sum_{j=1}^{2n^2t\gamma}\ \sum_{v\in\{(2n\rho\sigma,\infty)\cap\mathbb{Z}\}\times\{j\}} \mathbb{P}[H(v) > v(1)] \\
&\leq 2(2n^2t\gamma)\sum_{i\geq 1}\mathbb{P}[H > i + 2n\rho\gamma] \\
&\leq 2(2n^2t\gamma)\sum_{i\geq 1}\frac{\mathbb{E}[H^6]}{(i+2n\rho\sigma)^6} \\
&\leq 2(2n^2t\gamma)\sum_{i\geq 1}\frac{\mathbb{E}[H^6]}{i^3(2n\rho\sigma)^3} = \frac{C(t,\rho)}{n} \to 0 \text{ as } n\to\infty.
\end{aligned}$$

This completes the proof. □

Lemma 3.4.4. *Let $x, y, x_1, \ldots, x_m$ be points in $\mathbb{Z}$ with $x < y$. Define $u = (x, 0)$ and $v = (y, 0)$. Consider the random times $(T_n)_{n\geq 1}, (\tau_n(u))_{n\geq 1}$ and $(\tau_n(v))_{n\geq 1}$ as introduced in Corollary 2.2.1 for the points $(x_1, 0),\ldots,(x_m, 0)$,u and v. Put $T_0 = \tau_0(u) = \tau_0(v) = 0$ and let $\widetilde{\pi}^u$ and $\widetilde{\pi}^v$ be the linear interpolations of $(X_{\tau_n(u)}(u))_{n\geq 0}$ and $(X_{\tau_n(v)}(v))_{n\geq 0}$ respectively. Then for $\rho > 0$ and the stopping time*

$$\nu_{x,y,\rho^+} := \inf\{s \geq 0; \widetilde{\pi}^x(s) - \widetilde{\pi}^y(s) \geq n\rho\sigma\},$$

there exists a constant $C(t, \rho)$ depending only on t and ρ such that for all n large enough we have that

$$\mathbb{P}\big[\nu_{x,y,\rho^+} < \nu_{x,y} \wedge (n^2t\gamma)\big] < \frac{C(t,\rho)}{n}$$

where $\nu_{x,y}$ is the first time that $\widetilde{\pi}^x$ and $\widetilde{\pi}^y$ coalesce.

Proof. Here we follows ideas used in the proof of Lemma 2.4 in [12] and Lemma 3.2 in [6]. For $l \in \mathbb{Z}$ consider $u_0 = (0, 0)$, $u_l = (l, 0)$ and the random times $(\tau_n(u_0))_{n\geq 1}, (\tau_n(u_l))_{n\geq 1}$ as introduced in Corollary 2.2.1 for the points u_0, u_l. Now define the following random walk

$$Y_0^l := l, Y_n^l := X_{\tau_n(u_0)}(u_0)(1) - X_{\tau_n(u_l)}(u_l)(1) \text{ for } n \geq 1.$$

Let $B^l(x, t)$ be the set of trajectories that remain in the interval $[l - x, l + x]$ during the time $[0, t]$. By the independence of the increments which implies the strong Markov property, for any $x_1, y_1 \in \mathbb{R}$ we have that

$$\mathbb{P}(\nu_{x_1,y_1} > n^2\gamma t) \geq \mathbb{P}(\nu_{x_1,y_1,\rho^+} < n^2\gamma t \wedge \nu_{x_1,y_1}) \inf_{l\in\mathbb{Z}} \mathbb{P}(Y^l \in B^l(n\sigma\rho, n^2\gamma t)).$$

Note that

$$\inf_{l\in\mathbb{Z}} \mathbb{P}\big(Y^l \in B^l(n\sigma\rho, n^2\gamma t)\big) = 1 - \sup_{l\in\mathbb{Z}} \mathbb{P}\Big(\sup_{i\leq n^2\gamma t} |Y_i^l - l| \geq n\sigma\rho\Big).$$

Now

$$\limsup_{n\to\infty}\sup_{l\in\mathbb{Z}}\mathbb{P}\Big(\sup_{i\le n^2\gamma t}|Y_i^l - l| > n\sigma\rho\Big)\,,$$

is bounded above by

$$\begin{aligned}\limsup_{n\to\infty}\sup_{l\in\mathbb{Z}}\mathbb{P}\Big(&\sup_{i\le n^2\gamma t}|X_{\tau_i(u_0)}(u_0)(1)| + |X_{\tau_i(u_l)}(u_l)(1) - l| > \frac{n\sigma\rho}{2}\Big)\\ &\le 2\limsup_{n\to\infty}\sup_{l\in\mathbb{Z}}\mathbb{P}\Big(\sup_{i\le n^2\gamma t}|X_{\tau_i(u_0)}(u_0)(1)| > \frac{n\sigma\rho}{4}\Big)\\ &\le 4\mathbb{P}\Big(N > \frac{\rho}{4\sqrt{t}}\Big) = 4e^{-\frac{\rho^2}{32\,t}}\,,\end{aligned}$$

where N is a standard normal random variable and the last inequality is a consequence of Donsker's Theorem. Hence

$$\inf_{l\in\mathbb{Z}}\mathbb{P}\big(Y^l\in B^l(n\sigma\rho, n^2\gamma t)\big)$$

is bounded from below by a constant that depends only on t and ρ. So using Proposition 2.3.1 we obtain a constant $\tilde{C}(t,\rho)$ such that

$$\mathbb{P}(\nu_{x_1,y_1,\rho^+} < n^2\gamma t\wedge \nu_{x_1,y_1}) \le \frac{\mathbb{P}(\nu_{x_1,y_1} > n^2\gamma t)}{\inf_{l\in\mathbb{Z}}\mathbb{P}\big(Y^l\in B^l(n\sigma\rho, n^2\gamma t)\big)} \le \frac{\tilde{C}(t,\rho)\,|y_1 - x_1|}{n}\,.$$

RIWRITE THIS !!!

Now to get Lemma 3.4.4 we will condition on the first integer time k such that $\widetilde{\pi}^x(k) - \widetilde{\pi}^y(k)$ is bigger than zero.

$$\begin{aligned}&\mathbb{P}\big[\nu_{x,y,\rho^+} < \nu_{x,y}\wedge(n^2t\gamma)\big]\\ &= \sum_{i=0}^{\infty}\sum_{k=1}^{\lfloor n^2t\gamma\rfloor}\mathbb{P}[\nu_{x,y,0^+} = k, \widetilde{\pi}^x(k) - \widetilde{\pi}^y(k) = i]\mathbb{P}\big[\nu_{0,i,\rho^+} < \nu_{0,i}\wedge(n^2t\gamma - k)\big]\\ &\le \sum_{i=0}^{\infty}\sum_{k=1}^{\lfloor n^2t\gamma\rfloor}\mathbb{P}[\nu_{x,y,0^+} = k, \widetilde{\pi}^x(k) - \widetilde{\pi}^y(k) = i]\frac{\tilde{C}(t,\rho)i}{n}\\ &= \frac{\tilde{C}(t,\rho)}{n}\sum_{i=0}^{\infty}\sum_{k=1}^{\lfloor n^2t\gamma\rfloor}\mathbb{P}[\nu_{x,y,0^+} = k, \widetilde{\pi}^x(k) - \widetilde{\pi}^y(k) = i]i\\ &\le \frac{\tilde{C}(t,\rho)}{n}\mathbb{E}\big[\widetilde{\pi}^x(\nu_{x,y,0^+}) - \widetilde{\pi}^y(\nu_{x,y,0^+})\big] \le \frac{C(t,\rho)}{n}.\end{aligned}$$

Where the last inequality we get using item (ii) of Lemma 2.3.1. □

Remark 3.4.1. *The paths $\widetilde{\pi}^u$ and $\widetilde{\pi}^v$ in the statement of Lemma 3.4.4 are not paths of the GRDF. They are obtained from linear interpolation only on the points visited by the GRDF paths on the renewal times.*

Proof of Proposition 3.4.1. Let $\pi_1, \pi_2, \pi_3, \pi_4$ be the paths that start in $5\lfloor n\rho\sigma\rfloor$, $9\lfloor n\rho\sigma\rfloor$, $13\lfloor n\rho\sigma\rfloor$ and $17\lfloor n\rho\sigma\rfloor$ respectively at time zero. Let us denote the event that π_i stay within a distance $n\rho\sigma$ of $\pi_i(0)$ until time $2tn^2\gamma$ by $B_i^{n,t}$ for $i = 1, \dots, 4$, see Figure 9 below. From the invariance principle we have that

$$\lim_n \mathbb{P}[(B_i^{n,t})^c] = \mathbb{P}[\sup_{s\in[0,t]} |\mathcal{B}_s| > \rho] \leq 4e^{-\frac{\rho^2}{2t}}$$

for all $i = 1, \dots, 4$. Then

$$\frac{1}{t} \lim_{n\to\infty} \mathbb{P}\Big[(B_i^{n,t})^c\Big] \to 0 \text{ as } t \to 0^+. \tag{3.4.1}$$

See that

$$\begin{aligned}
&\lim_{t\to 0^+} \frac{1}{t} \limsup_{n\to\infty} \mathbb{P}\Big[A^+_{\mathcal{X}_n}(0,0;\rho,t)\Big] \\
&= \lim_{t\to 0^+} \frac{1}{t} \limsup_{n\to\infty} \mathbb{P}\Big[A^+_{\mathcal{X}}(0,0;\rho n\sigma, tn^2\gamma)\Big] \\
&\leq \lim_{t\to 0^+} \frac{1}{t} \limsup_{n\to\infty} \mathbb{P}\Big[D(n\rho\sigma, tn^2\gamma)\Big] + 4 \lim_{t\to 0^+} \frac{1}{t} \lim_{n\to\infty} \mathbb{P}\Big[(B_1^{n,t})^c\Big] + \\
&+ \lim_{t\to 0^+} \frac{1}{t} \limsup_{n\to\infty} \mathbb{P}\Big[A^+_{\mathcal{X}}(0,0;\rho n\sigma, tn^2\gamma), \cap_{i=1}^4 B_i^{n,t}, \big(D(n\rho\sigma, tn^2\gamma)\big)^c\Big].
\end{aligned}$$

By Lemma 3.4.3 we have that

$$\limsup_{n\to\infty} \mathbb{P}\Big[D(n\rho\sigma, tn^2\gamma)\Big] = 0\,, \quad \text{for every } t > 0$$

and by (3.4.1)

$$\lim_{t\to 0^+} \frac{1}{t} \lim_{n\to\infty} \mathbb{P}\Big[(B_1^{n,t})^c\Big] = 0\,.$$

So we only have to prove, see Figure 9, that for every $t > 0$

$$\limsup_{n\to\infty} \mathbb{P}\Big[A^+_{\mathcal{X}}(0,0;\rho n\sigma, tn^2\gamma), \cap_{i=1}^4 B_i^{n,t}, (D(n\rho\sigma, tn^2\gamma))^c\Big] = 0\,. \tag{3.4.2}$$

Fix some $(x,m) \in R(0,0;2n\rho\sigma, 2n^2t\gamma)$ and take $(T_i)_{i\geq 1}$ renewal times introduced in the Corollary 2.2.2 for the points $(5\lfloor n\rho\sigma\rfloor, 0)$,$(9\lfloor n\rho\sigma\rfloor, 0)$,$(13\lfloor n\rho\sigma\rfloor, 0)$, $(17\lfloor n\rho\sigma\rfloor, 0)$ and the (x,m). We will denote by $\big(Y_i^{(x,m)}\big)_{i\geq 0}$ the random walks built as the first component of the path $\pi^{(x,m)}$ on the renewal times $(T_i)_{i\geq 1}$. Take the stopping times $\nu_j^{(x,m)}$ for $j = 1, \dots, 5$ as the first time that $\big(Y_i^{(x,m)}\big)_{i\geq 0}$ exceeds $(4j-1)\lfloor n\rho\sigma\rfloor$; and $\nu^{(x,m)}$ the first time that $\pi^{(x,m)}$ exceeds $20n\rho\sigma$. Then

$$\begin{aligned}
&\mathbb{P}\big[\nu^{(x,m)} < 4n^2t\gamma, \cap_{i=1}^4 B_i^{n,t}\big] \\
&\leq \mathbb{P}\big[T_{\nu_5^{(x,m)}} < 4n^2t\gamma, \cap_{i=1}^4 B_i^{n,t}\big] + \mathbb{P}\big[\nu^{(x,m)} < 4n^2t\gamma, T_{\nu_5^{(x,m)}} \geq 4n^2t\gamma\big].
\end{aligned} \tag{3.4.3}$$

Note that on the event $\big\{\nu^{(x,m)} < 4n^2\gamma t, T_{\nu_5^{(x,m)}} \geq 4n^2t\gamma\big\}$ the path $\pi^{(x,m)}$ cross the interval $\big(19\lfloor n\rho\sigma\rfloor, 20n\rho\sigma\big)$ without renewal before time $n^2t\gamma$. Because the displacement between consecutive renewal times is bounded by some random variable Z with finite moments, and up to time $n^2\gamma t$ the number of renewals is bounded by $n^2\gamma t$ we have that

$$\mathbb{P}\big[\nu^{(x,m)} < 4n^2\gamma t, T_{\nu_5^{(x,m)}} \geq 4n^2t\gamma\big] \leq n^2t\gamma\mathbb{P}\big[Z > n\rho\sigma\big] \leq \frac{n^2t\gamma\mathbb{E}[Z^6]}{(n\rho\sigma)^6} \leq \frac{C_1}{n^4}. \tag{3.4.4}$$

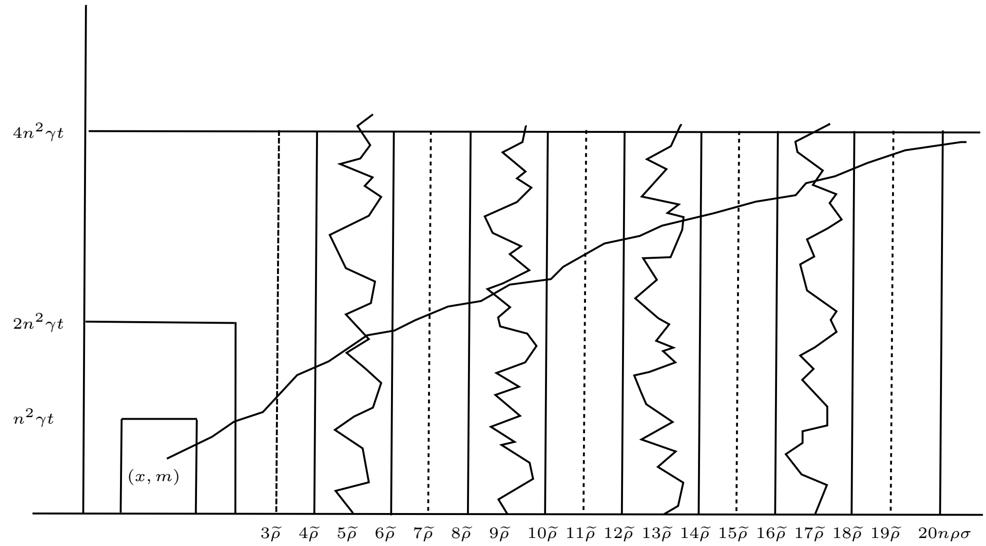

Figure 3.2: Realization of $A^+_{\mathcal{X}}(0,0;\rho n\sigma, tn^2\gamma)\cap\cap_{i=1}^4 B_i^{n,t}$ where $A^+_{\mathcal{X}}(0,0;\rho n\sigma, tn^2\gamma)$ occurs because the path $\pi^{(x,m)}$, for some $(x,m)\in R(0,0;\rho n\sigma,tn^2\gamma)$, touchs the right boundary of the rectangle $R(0,0;20\rho n\sigma,4tn^2\gamma)$. Notation: $\tilde\rho=\lfloor n\sigma\rho\rfloor$.

We also have

$$\begin{aligned}
&\mathbb{P}\Big[T_{\nu_5^{(x,m)}}<4n^2t\gamma,\cap_{i=1}^4 B_i^{n,t}\Big]\\
&\le\mathbb{P}\Big[Y^{(x,m)}_{\nu_j^{(x,m)}}\le\big(4j-\frac{1}{2}\big)\lfloor n\rho\sigma\rfloor,\ j=1,...5,\ T_{\nu_5^{(x,m)}}<4n^2t\gamma,\cap_{i=1}^4 B_i^{n,t}\Big]\\
&+\sum_{j=1}^5\mathbb{P}\Big[Y^{(x,m)}_{\nu_j^{(x,m)}}>\big(4j-\frac{1}{2}\big)\lfloor n\rho\sigma\rfloor\Big]\\
&\le\mathbb{P}\Big[Y^{(x,m)}_{\nu_j^{(x,m)}}\le\big(4j-\frac{1}{2}\big)\lfloor n\rho\sigma\rfloor,\ j=1,...5,\ T_{\nu_5^{(x,m)}}<4n^2t\gamma,\cap_{i=1}^4 B_i^{n,t}\Big]\\
&+5\sup_{x\in\mathbb{Z}_-}\mathbb{P}\Big[Y^{(x,m)}_{\nu_+^x}>\frac{\lfloor n\rho\sigma\rfloor}{2}\Big].
\end{aligned}\tag{3.4.5}$$

By the Lemma 3.4.2 and Corollary 2.2.2 there exists a constant C_2 such that

$$\sup_{x\in\mathbb{Z}_-}\mathbb{P}\Big[Y^{(x,m)}_{\nu_+^x}>\frac{\lfloor n\rho\sigma\rfloor}{2}\Big]\le\frac{C_2}{n^4}.$$

Using the strong Markov property and Lemma 3.4.4 we get a constant C_3 such that

$$\begin{aligned}
&\mathbb{P}\Big[Y^{(x,m)}_{\nu_j^{(x,m)}}\le\big(4j-\frac{1}{2}\big)\lfloor n\rho\sigma\rfloor,\ j=1,...5,\ T_{\nu_5^{(x,m)}}<4n^2t\gamma,\cap_{i=1}^4 B_i^{n,t}\Big]\\
&\le\mathbb{P}\Big[\nu^+_{x,y,\rho}<\nu_{x,y}\wedge(n^2t\gamma)\Big]^4\le\frac{C_3}{n^4}.
\end{aligned}$$

Hence by (3.4.5)

$$\mathbb{P}\Big[T_{\nu_5^{(x,m)}}<4n^2t\gamma,\cap_i^4 B_i^{n,t}\Big]\le\frac{C_3}{n^4}+\frac{5C_2}{n^4}.\tag{3.4.6}$$

Now we can go back to (3.4.3), use (3.4.4), (3.4.5) and (3.4.6) to conclude that

$$\mathbb{P}\big[\nu^{(x,m)}<4n^2t\gamma,\cap_{i=1}^4 B_i^{n,t}\big]\le\frac{(5C_1+C_2+C_3)}{n^4}.$$

Therefore we can estimate the probability in (3.4.2) as

$$\begin{aligned}&\mathbb{P}\Big[A^+_{\mathcal{X}}(0,0;\rho n\sigma, tn^2\gamma), \cap_{i=1}^4 B_i^{n,t}, \{D(n\rho\sigma, n^2t\gamma)\}^c\Big]\\&\leq \mathbb{P}\Big[\exists(x,m) \in R(0,0;2n\rho\sigma, 2n^2t\gamma); \nu^{(x,m)} < 4n^2t\gamma, \cap_{i=1}^4 B_i^{n,t}\Big].\end{aligned}$$

Since $R(0,0;2n\rho\sigma, 2n^2t\gamma)$ has $8t\rho\sigma\gamma n^3$ points we have that

$$\begin{aligned}&\limsup_{n\to\infty} \mathbb{P}\Big[A^+_{\mathcal{X}}(0,0;\rho n\sigma, tn^2\gamma), \cap_{i=1}^4 B_i^{n,t}, \{D(n\rho\sigma, n^2t\gamma)\}^c\Big]\\&\leq \limsup_{n\to\infty}(8t\rho\sigma\gamma n^3)\mathbb{P}\big[\nu^{(x,m)} < 4n^2t\gamma, \cap_{i=1}^4 B_i^{n,t}\big]\\&\leq \limsup_{n\to\infty}(8t\rho\sigma\gamma n^3)\frac{(5C_1 + C_2 + C_3)}{n^4} = 0.\end{aligned}$$

□

BIBLIOGRAPHY

[1] Arratia, R. *Coalescing Brownian motions on the line.* PhD thesis, University of Wisconsin, Madison., 1979.

[2] Arratia, R. Coalescing brownian motions and the voter model on $\mathbb{Z}$. unpublished partial manuscript (circa 1981). *Unpublished partial manuscript (circa 1981), available at* `rarratia@math.usc.edu` (1981).

[3] Baccelli, F., and Bordenave, C. The radial spanning tree of a poisson point process. *The Annals of Applied Probability* (2007), 305–359.

[4] Belhaouari, S., Mountford, T., Sun, R., and Valle, G. Convergence and sharp results for the voter model interfaces. *Electronic Journal of Probability 11* (2006), 279–296.

[5] Coletti, C., Fontes, L. R., and Dias, E. S. Scaling limit for a drainage network model. *Journal of Applied Probability 46*, 4 (2009), 1184–1197.

[6] Coletti, C., and Valle, G. Convergence to the brownian web for a generalization of the drainage network model. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques 50*, 3 (2014), 899–919.

[7] Ferrari, P., Fontes, L. R., and Wu, X.-Y. Two-dimensional poisson trees converge to the brownian web. *Annales de l'Institut Henri Poincaré. B, Probability and Statistics 41* (2014), 851–858.

[8] Fontes, L., Isopi, M., Newman, C., and Ravishankar, K. The brownian web. *Proceedings of the National Academy of Sciences 99*, 25 (2002), 15888–15893.

[9] Fontes, L., Isopi, M., Newman, C., and Ravishankar, K. Coarsening, nucleation, and the marked brownian web. In *Annales de l'Institut Henri Poincare (B) Probability and Statistics* (2006), vol. 42, Elsevier, pp. 37–60.

[10] Fontes, L. R., Isopi, M., Newman, C. M., and Ravishankar, K. The brownian web: characterization and convergence. *The Annals of Probability 32*, 4 (2004), 2857–2883.

[11] Fontes, L. R., Valencia, L., and Valle, G. Scaling limit of the radial poissonian web. *Electronic Journal of Probability 20* (2015).

[12] Newman, C. M., Ravishankar, K., and Sun, R. Convergence of coalescing nonsimple random walks to the brownian web. *Electronic Journal of Probability 10* (2005), 21–60.

[13] Roy, R., Saha, K., and Sarkar, A. Random directed forest and the brownian web. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques 52*, 3 (2016), 1106–1143.

[14] Schertzer, E., Sun, R., and Swart, J. M. The brownian web, the brownian net, and their universality. *arXiv preprint arXiv:1506.00724* (2015).

[15] Tóth, B., and Werner, W. The true self-repelling motion. *Probability Theory and Related Fields 111*, 3 (1998), 375–452.