METASTABILITY FOR THE CONTACT PROCESS WITH TWO SPECIES AND PRIORITY

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Metastability for the contact process with two species and priority.

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O processo de contato um processo estocstico que pode ser interpretado como a evoluo temporal de uma certa populao. Neste trabalho estudamos uma variante do processo de contato clssico onde temos dois tipos de indivduos na populao, os indivduos de tipo 1 e os de tipo 2. Para este caso provamos que o modelo apresenta metaestabilidade. Um sistema considerado em uma situao metaestvel si se comporta como uma distribuio de falso equilbrio durante um tempo grande at que de forma abrupta alcana o verdadeiro equilbrio. Na abordagem utilizada para obter a metaestabilidade so principais os conceitos de renormalizao e conhecimentos de percolao orientada.

Palavras chaves: Processo de contato, percolao.

Metastability for the contact process with two species and priority.

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The contact process is a stochastic process that can be interpreted as the time evolution of a certain population. This process was given by Ted Harris in 1974. In this work, we study a variation of the classic contact where there are two types of individuals, individuals of type 1 and individuals of type 2. In this case, we prove that this process presents the metastability phenomenon. A system is considered in a metastable situation if it behaves as in a false equilibrium distribution for a random long time until abruptly it gets the true equilibrium. In the approach to obtain the metastability for this process we use of concept of renormalization and results of oriented percolation.

Keywords: Contact process, percolation.

Contents

Introduction

The phenomenon of metastability has been widely study in physics and mathematics. A system is considered in a metastable situation if it behaves as in a *false equilibrium* distribution for a long random time until abruptly it gets to the true equilibrium. Classical examples of this phenomenon include the behavior of supercooled vapours and liquids, and supersaturated vapours and solutions. For a detailed discussion on metastability in stochastic processes and references, see the monographs [4] and [17].

A specific stochastic process that fits into this situation is the *contact process*, introduced by Harris in [11]. It is a simple model for the spread of an infection, where individuals are identified with the vertices of a given graph which we may take as $\mathbb{Z}^d$. In this model every infected individual can propagate the infection to some neighbor at rate λ and it becomes healthy at rate 1. The contact process can also be interpreted as the time evolution of a certain population, where a site is now "occupied" (in correspondence to "infected") or "empty"(in correspondence to"healthy").

An important characteristic of the Harris contact process is that it presents a dynamical phase transition, namely: there exists a critical value λ_c for the infection rate such that if λ is larger than λ_c there is a non trivial invariant measure μ different from $\delta_\emptyset$. On the other hand, when the process is restricted to a finite volume it is a finite Markov chain and $\delta_\emptyset$ is the only equilibrium state. Nevertheless, for suitable initial conditions, the restriction of the non-trivial invariant measure μ to this finite volume behaves as a metastable state as described above. This was first proved in [5] for the one-dimensional case and λ sufficiently large. In this paper, the authors introduced a pathwise point of view for the study of metastability in stochastic dynamics. That is, they proposed to describe the dynamical phenomenon of metastability through the convergence of suitably rescaled transition time to an exponential distribution and with a stabilization of suitably rescaled time averages along the evolution around a non equilibrium distribution. This last convergence is named thermalization property. The results in [5] were extended in [18] to the whole supercritical region. A different proof of the asymptotic exponential time was proved in [9], which also describes the asymptotic behavior of the logarithm of the time of extinction. These last results were extended for dimension $d \geq 2$ in [14] and [15], respectively. The thermalization property for the contact process in dimension $d \geq 2$ was proved in [19].

There are some examples of processes inspired by the contact process that try to describe what happens if the population is not homogeneous, in the sense that some individuals have different characteristics. An example is the process introduced in [10] in which every site in $\mathbb{Z}$ can be occupied by particles of type 1 or 2, but the particles of type 1 have priority throughout the environment.

The aim of this work is the study of a metastable phenomenon for a stochastic process that can be interpreted as the time evolution of a population, which has two different species and each of them has a favorable region in the environment. In the process we present, the priority is no longer spatially homogeneous. Specifically, we will consider $\mathbb{Z}$ and a layer of $\mathbb{Z}^2$ as the environments, which will be denoted by S, and we choose a partition $(\mathrm{R}_1, \mathrm{R}_2)$ of the environment, such that the particles of type 1 have priority in R_1 and the particles of type 2 in R_2.

The process we are interested is a continuous time Markov process with state space $\{0,1,2\}^S$ and we denote it by $\{\zeta_t\}_t$. If the site x is occupied at time t by a particle of type i (i=1,2) we set $\zeta_t(x) = i$, and if the site x is empty at time t we set $\zeta_t(x) = 0$. We denote the flips rates at x in a configuration $\zeta \in \{0,1,2\}^S$ by $c(x, \zeta, \cdot)$ and we define them as follows

$$
\begin{aligned}
&c(x,\zeta,1\to 0) = c(x,\zeta,2\to 0) = 1,\\
&c(x,\zeta,0\to i) = \lambda \sum_{y:\ 0<|x-y|\leq R} \mathbb{1}_{\zeta(y)=i}, i = 1,2,\\
&c(x,\zeta,2\to 1) = \lambda \sum_{y:\ 0<|x-y|\leq R} \mathbb{1}_{\zeta(y)=1}\mathbb{1}_{\{x\in \mathrm{R}_1\}},\\
&c(x,\zeta,1\to 2) = \lambda \sum_{y:\ 0<|x-y|\leq R} \mathbb{1}_{\zeta(y)=1}\mathbb{1}_{\{x\in \mathrm{R}_2\}},
\end{aligned}
$$

where R is a scalar larger than 1, it is known as the range of the process. During the thesis we refer to the process with those flips rates as the *contact process with two types of particles and priority.*

We prove that the process, when beginning with full occupancy of particles of type 1 in

R_1 and particles of type 2 in R_2, presents the metastable phenomenon. More precisely if the dynamics is restricted to a finite box with dimensions depending on N, the time when one of the two families of particles becomes extinct, when properly rescaled, converges to the exponential distribution as N tends to infinite. In the case that $S = \mathbb{Z}$ we also prove a result which gives information on the asymptotic order of magnitude of this time (for the limit in N). Together with known results, this implies the existence of two metastable regimes for this process: one with both species and the standard one for the contact process.

This work is organized in four chapters. Chapter 1 contains five sections. In Section 1.1 we introduce the Harris graph and some notations for the contact process that will be used during the work. In Section 1.2 we give a graphical definition of our process when it is restricted to a spatial box. In Section 1.3 we recall some results on oriented percolation that will be used in the next chapters. In Section 1.4 we review the definition of Mountford-Sweet renormalization. Finally, in Section 1.5 we also review the definition of Benzuidenhout-Grimmett renormalization.

In Chapter 2 we shall examine the case when $S = \mathbb{Z}$, $R_1 = (-\infty, 0]$, $R_2 = [1, \infty)$, initial population $\mathbb{1}_{(-\infty,0]} + 2\mathbb{1}_{[1,\infty)}$ and $R \geq 1$. In Section 2.1, we define *barriers* in a finite interval; this is a central tool in the development of the next sections. In Section 2.2, we present a result about the metastability for the contact process in dimension 1 with range $R > 1$. In Section 2.3, we prove that the time when one of the two families becomes extinct in the interval converges to an exponential distribution as the length of the interval tends to infinite. In Section 2.4, we prove the convergence in probability of the logarithm of this time divided by the length of the interval to a positive constant.

In Chapter 3 we consider $S = [-L, L] \times \mathbb{Z}$, $R = 1$, $R_1 = [-L, L] \times (-\infty, 0]$ and $R_2 = [-L, L] \times [1, \infty)$ and initial population $\mathbb{1}_{[-L,L]\times(-\infty,0]} + 2\mathbb{1}_{[-L,L]\times[1,\infty)}$, where L is an arbitrary but fixed positive number. In Section 3.1 we define an extension of Mountford-Sweet renormalization for the layer. In Section 3.2 we prove that the contact process with two types of particles and priority in the layer presents a metastable behavior.

Chapter 4 has three sections. In Section 4.2 we prove that for the process treated in Chapter 2 there exists an invariant measure that gives total measure to the configurations with infinity particles of type 1 and infinity particles of type 2. In Section 4.3 we choose $S = \mathbb{Z}$, $R_1 = (-\infty, 0]$, $R_2 = [1, \infty)$, initial population $2\mathbb{1}_{(-\infty,0]} + \mathbb{1}_{[1,\infty)}$ and $R = 1$. For this case we prove that in an interval of length N the time when one of the species win the competition is at most linear on N. This time is pretty small in contrast with the equivalent time when each species begins in their favorable regions.

Chapter 1

Preliminaries

1.1 Contact Process

In this section, we recall the Harris construction introduced in [11]. Using this construction, we define the classic contact process.

In order to define the contact process with range R, we consider a collection of independent Poisson processes on $[0, \infty)$

$$\begin{aligned} &\{P^x\}_{x\in\mathbb{Z}^d} \text{ with rate } 1, \\ &\{P^{x\to y}\}_{\{x,y\in\mathbb{Z}^d:\ 0<|x-y|\leq R\}} \text{ with rate } \lambda, \end{aligned} \tag{1.1.1}$$

where $R \in \mathbb{N}$. Graphically, we identify the realization of the process P^x at a point $(x, t) \in \mathbb{Z}^d \times [0, +\infty)$, with a cross mark, and a realization of the process $P^{x\to y}$ at the same point with an arrow following the direction x to y. We denote by $\mathcal{H}$ a realization of all Poisson processes, this is a *Harris construction* (see Figure 1.1). Whenever we refer to the probability of events involving the contact process, we implicitly assume that $\mathbb{P}$ is a probability under which $\mathcal{H}$ has the law defined above; sometimes, we write $\mathbb{P}_\lambda$ to be explicit about the value of the infection rate. Given $(x, t) \in \mathbb{Z}^d \times [0, \infty)$ we define $\Theta_{(x,t)}(\mathcal{H})$ as the shifting of the Harris construction $\mathcal{H}$, where (x, t) becomes the origin.

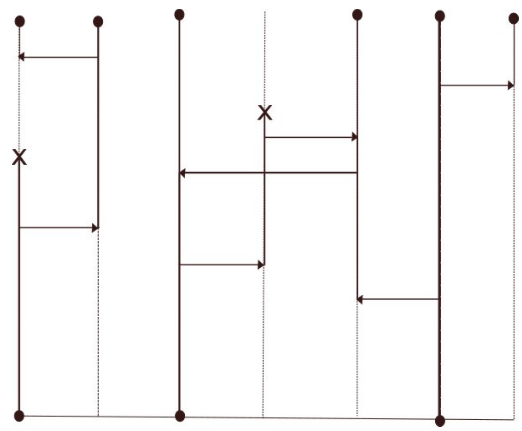

Figure 1.1: An example of a Harris construction for $R > 1$.

A *path* on $\mathcal{H}$ is an oriented path following the positive direction of time t which passes along the arrows in the direction of them but does not pass through any cross mark. More precisely, we denote $(x, s) \to (y, t)$, with $0 < s < t$, if there exists a càdlàg $\gamma : [s, t] \to \mathbb{Z}^d$ such that:

- $\gamma(s) = x, \gamma(t) = y$,
- $\gamma(z) \neq \gamma(z-)$ only if $z \in P^{\gamma(z-)\to\gamma(z)}$,
- $\forall z \in [s, t], z \notin P^{\gamma(z)}$.

For A and B subsets of $\mathbb{Z}^d$ and $0 \leq s < t$ we say that $A \times \{s\}$ is connected with $B \times \{t\}$ if there are $x \in A$ and $y \in B$ such that $(x, s) \to (y, t)$, and we denote by $A \times \{s\} \to B \times \{t\}$.

To simplify the notation, throughout all the work for every spatial set $I \subset \mathbb{R}^d$ we identify $I \cap \mathbb{Z}^d$. Also, we identify every configuration $\eta \in \{0, 1\}^{\mathbb{Z}^d}$ with the subset $\{x \in \mathbb{Z}^d : \eta(x) = 1\}$.

Given a Harris construction and a subset A of $\mathbb{Z}^d$, we can define the contact process beginning at time s, with initial configuration A as follows

$$_{(s)}\xi^A(t) = \{x : \text{ exists } y \in A \text{ such that } (y, s) \to (x, t)\}, \tag{1.1.2}$$

in the case $s = 0$ we omit the underscript (0). We also define the time of extinction as follows

$$T^A = \inf\{t > 0 : \xi_t^A = \emptyset\}. \tag{1.1.3}$$

Let A and C be subsets of $\mathbb{Z}^d$ such that $A \subset C$, we define the contact process restricted to C with initial configuration A as

$$\xi_C^A(t) = \{x : \text{ exists } y \in A \text{ such that } (y, 0) \to (x, t) \text{ inside } C\}. \tag{1.1.4}$$

In the special case that $C = [1, N]^d$ we use the notation $\xi_N^A(t)$. For this process, we define the time of extinction as follows

$$T_N^A = \inf\{t > 0 : \xi_N^A(t) = \emptyset\}. \tag{1.1.5}$$

Special notation: $\xi^{\mathbb{1}}(t)$ for initial configuration $\mathbb{Z}^d$; $\xi^x(t)$ and T^x for initial configuration $\{x\}$; $\xi_N^{\mathbf{1}}(t)$, $T_N^{\mathbf{1}}$ for initial configuration full occupancy in $[1, N]^d$, $\xi_N^x(t)$, T_N^x for initial configuration $\{x\}$.

For a time t and a set A, we define the dual contact process at time $s \in [0, t]$, with initial configuration A as

$$\tilde{\xi}^{A,t}(s) = \{x : \text{ exists } y \in A \text{ such that } (x, t - s) \to (y, t)\}. \tag{1.1.6}$$

We observe that the process $\{\tilde{\xi}^{A,t}(s)\}_{0 \le s \le t}$ has the same law as the contact process until time t with initial configuration A.

As we mentioned in the introduction, the contact process presents a phase transition with respect to the rate of infection λ: there exists a critical parameter $\lambda_c = \lambda_c(d, R)$ defined as follows

$$\lambda_c = \inf\{\lambda : \mathbb{P}_\lambda(T^0 = \infty) > 0\}.$$

For all $\lambda > \lambda_c$ all invariant measure of the contact process is a convex combination of $\delta_\emptyset$ and a non trivial measure μ_λ. During all our work we are considering $\lambda > \lambda_c$.

For the contact process in dimension 1 and finite range with initial configuration $(-\infty, 0]$, we denote the rightmost infected particle by

$$r_t^{(-\infty,0]} = \max\{x : \xi^{(-\infty,0]}(t)(x) = 1\} \tag{1.1.7}$$

and the leftmost infected particle connected with $[0, \infty)$ at time t as

$$l_t^{[0,\infty)} = \min\{x : \xi^{[0,\infty)}(t)(x) = 1\}. \tag{1.1.8}$$

By the symmetry of the Harris construction, we have that $r_t^{(-\infty,0]}$ has the same law as $-l_t^{[0,\infty)}$ for all t. In [12] is proved that for $R = 1$ there exists $\alpha > 0$ such that

$$\frac{r_t^{(-\infty,0]}}{t} \underset{t\to\infty}{\longrightarrow} \alpha \text{ almost surely.} \tag{1.1.9}$$

This result is obtained using the Subadditive Ergodic Theorem and monotonicity arguments that can be adapted for the case $R > 1$.

From the Harris construction we observe that if $A \subset B$ then $\xi_t^A \subset \xi_t^B$ for all t. This property is called attractiveness.

1.2 Graphical construction of the contact process with two types of particles and priority

In this section we give a graphical construction of the contact process with two types of particles and priority. This graphical definition uses the Harris construction and gives a natural coupling with the usual contact process.

Let A and B be two disjoint subsets of $[-N+1, N]^d$, we denote by $\{\zeta_t^{A,B,N}\}_t$ the contact process restricted to the set $[-N+1, N]^d$ with two types of particles, initial configuration $\mathbb{1}_A + 2\mathbb{1}_B$ and the particles of type 1 having priority in $[-N+1, N]^{d-1} \times [-N+1, 0]$ and the type 2 in $[-N+1, N]^{d-1} \times [1, N]$. In this case, it is simple to state the definition of the process in terms of a Harris construction, since we are dealing with a càdlàg stochastic process with jumps only in the times of the Poisson processes $\{P^x\}_{x\in[-N+1,N]^d}$ or $\{P^{y\to x}\}_{\{y,x\in[-N+1,N]^d:\ 0<|x-y|\le R\}}$. Let t be one of those times, two scenarios are possible:

(1) $t \in P^x$ for some x. In this case, x is empty at this time ($\zeta_t^{A,B,N}(x) = 0$);

(2) $t \in P^{y\to x}$ for some x and y. If x is occupied by a particle of type i ($i = 1, 2$) and x is in the region of priority of this kind of particles, then nothing changes at x. Out of this situation, x become occupied by the type of particle that is in y ($\zeta_t^{A,B,N}(x) = \zeta_t^{A,B,N}(y)$).

We are interested in the study of the time when one of the families became extinct and we denote this time by $\tau_N^{A,B}$. More precisely

$$\tau_N^{A,B} = \inf\{t : \zeta_t^{A,B,N}(x) \neq 1\ \forall\, x,\ \text{or}\ \zeta_t^{A,B,N}(x) \neq 2\ \forall\, x\}. \tag{1.2.1}$$

Special notation: $\zeta_t^{\mathbf{1},\mathbf{2},N}$ and $\tau_N^{\mathbf{1},\mathbf{2}}$, for initial configuration $\mathbb{1}_{[-N+1,N]^{d-1}\times[-N+1,0]} + 2\mathbb{1}_{[-N+1,N]^{d-1}\times[1,N]}$.

Remark 1.1. *Since the classic contact process and the contact process with two types of particles and priority are defined using the same Harris construction $\mathcal{H}$, both processes are defined in the same probability space. This coupling will be used in all the work.*

The next lemma follows from the definition of the process and it is very useful for proving a property called regeneration that will be explored in Section 2.2. The lemma states that if at time t a site x is occupied by a particle of type 1 (2), there exists a path of particles of type 1 (2) connecting the initial configuration with (x,t).

Lemma 1.1. *Let A and B be disjoint subsets of $[-N+1,N]^d$. Given the construction of the contact process with two types of particles and initial configuration $\zeta_0^{A,B,N} = \mathbb{1}_A + 2\mathbb{1}_B$, we have that*

$$\zeta_t^{A,B,N}(x) = 1 \Leftrightarrow \textit{There exists a path } \gamma \textit{ connecting } A \textit{ with } (x,t) \\ \textit{such that } \zeta_s^{A,B,N}(\gamma(s)) = 1, \quad \textit{for all } s,\ 0 \le s \le t,$$

where $x \in [-N+1,N]^d$ and $t > 0$.

Proof. ($\Leftarrow$) It is clear.

($\Rightarrow$) For a given realization of the Harris construction, let m be the number of the marks of the Poisson processes $\{P^x\}_{x\in[-N+1,N]}$ and $\{P^{x\to y}\}_{\{x,y\in[-N+1,N]:\ 0<|x-y|\le R\}}$ that appear before time t. Let t_i be the time of the i-th mark, and set $t_0 = 0$ and $t_{m+1} = t$. We will prove the statement by induction in i, $0 \le i \le m+1$. For time $t_0 = 0$ it is trivial. Now, suppose the statement holds for t_i and take y such that $\zeta_{t_{i+1}}^{A,B,N}(y) = 1$. We must find a path β connecting $A \times \{0\}$ with (y, t_{i+1}) with the desired properties. There are two possibilities:

1. $\zeta_{t_{i+1}}^{A,B,N}(y) = \zeta_{t_i}^{A,B,N}(y)$. In this case, by induction hypothesis there is γ connecting $A \times \{0\}$ with (y, t_i) satisfying $\zeta_s^{A,B,N}(\gamma(s)) = 1,\ \ 0 \le s \le t_i$. Define

$$\beta(s) = \begin{cases} \gamma(s) & 0 \le s < t_i, \\ y & t_i \le s \le t_{i+1}. \end{cases}$$

2. $\zeta_{t_{i+1}}^{A,B,N}(y) \neq \zeta_{t_i}^{A,B,N}(y)$. In this case, there is an integer $k \in [-R,R] \setminus \{0\}$ such that $t_i \in P^{y+k\to y}$ and $\zeta_{t_{i+1}}^{A,B,N}(y) = \zeta_{t_i}^{A,B,N}(y+k)$. By induction hypothesis, there is γ

connecting $A \times \{0\}$ with $(y+k, t_i)$ satisfying $\zeta_s^{A,B,N}(\gamma(s)) = 1,\ \ 0 \le s \le t_i$, and we define

$$\beta(s) = \begin{cases} \gamma(s) & 0 \le s < t_i \\ y+k & t_i \le s < t_{i+1} \\ y & s = t_{i+1}. \end{cases}$$

In each case above the path β satisfies

$$\zeta_s^{A,B,N}(\beta(s)) = 1,\ \ 0 \le s \le t_{i+1}.$$

Hence the proof of the lemma is now complete. □

1.3 Results on 1-dependent oriented percolation system with small closure

In the next section we recall the definition of an oriented percolation system introduced in [16] for the contact process in dimension 1 and range $R \ge 1$. This percolation system is an important tool for our results to the contact process with two types of particles and priority in dimension 1. Before the definition of this renormalization, we recall some notions and results of oriented percolation that we will need later .

Consider $\Lambda = \{(m,n) \in \mathbb{Z} \times \mathbb{Z}^+ : m+n \text{ is even }\}$, $\Omega = \{0,1\}^\Lambda$ and $\mathcal{F}$ the σ-algebra generated by the cylinder sets of Ω. Given $\Psi \in \Omega$, we say that two points (m,k), $(m',k') \in \Lambda$ with $k < k'$ are *connected by an open path* (*according to* Ψ) [1], if there exists a sequence $\{(m_i, n_i)\}_{0 \le i \le k'-k}$ such that

$$(m_0, n_0) = (m,k), \quad (m_{k'-k}, n_{k'-k}) = (m',k'), \quad |m_{i+1} - m_i| = 1, \quad n_i = k+i,$$

with $0 \le i \le k'-k-1$ and $\Psi(m_i, n_i) = 1$ for all i. If (m,k) and (m',k') are connected by an open path (according to Ψ), we write $(m,k) \rightsquigarrow (m',k')$ (according to Ψ).

Now, let A, B and C be subsets of Λ. We say that $A \times \{n\}$ is connected with $B \times \{n'\}$ inside C, if there are $m \in A$ and $m' \in B$ such that $(m,n) \rightsquigarrow (m',n')$ and all the edges of the path are in C. In this case, we write $A \times \{n\} \rightsquigarrow B \times \{n'\}$ inside C.

For $(y,k) \in \Lambda$ we denote the cluster beginning in (y,k) as follows

$$C_{(y,k)} = \{(x,n) : \text{ such that } n \geq k \text{ and } (x,n) \in \Lambda \text{ and } (y,k) \rightsquigarrow (x,n)\}.$$

Let C be a subset of $2\mathbb{Z}$, we denote the set of point connected at time n as

$$\Psi_n^C = \{x : \text{ exists } y \in C \text{ such that } (y,0) \rightsquigarrow (x,n)\}. \quad (1.3.1)$$

Also, we denote the rightmost particle connected with $(-\infty, 0]$ at time n as follows

$$\hat{r}_n = \max\{y : \exists \ m \leq 0, \text{ such that } (m,0) \rightsquigarrow (y,n)\}, \quad (1.3.2)$$

and the rightmost particle connected with a point $(x,0) \in \Lambda$ by

$$\hat{r}_n^{\{x\}} = \max\{y : (x,0) \rightsquigarrow (y,n)\}. \quad (1.3.3)$$

Given $k \geq 1$ and $\delta > 0$, $(\Omega, \mathcal{F}, \hat{\mathbb{P}})$ is a k-dependent oriented percolation system with closure below δ, if for all r positive

$$\hat{\mathbb{P}}(\Psi(m_i,n) = 0, \forall i \ \ 0 \leq i \leq r | \{\Psi(m,s) : (m,s) \in \Lambda, 0 \leq s < n\}) < \delta^r,$$

with $(m_i, n) \in \Lambda$ and $|m_i - m_j| > 2k$ for all $i \neq j$ and $1 \leq i,j \leq r$ (see [16], [1]).

Let Ψ and Ψ' be two elements of Ω, we say that $\Psi \leq \Psi'$ if $\Psi(m,n) \leq \Psi'(m,n)$, $\forall (m,n) \in \Lambda$. Also, we say that a subset A of $\{0,1\}^{\Lambda}$ is increasing if $\Psi \in A$ and $\Psi \leq \Psi'$, then $\Psi' \in A$. Let $\hat{\mathbb{P}}_1$ and $\hat{\mathbb{P}}_2$ be two measures on $\mathcal{F}$, we say that $\hat{\mathbb{P}}_1$ stochastically dominates $\hat{\mathbb{P}}_2$ if $\hat{\mathbb{P}}_1(A) \geq \hat{\mathbb{P}}_2(A)$ for all A increasing in $\mathcal{F}$.

Let $\hat{\mathbb{P}}_p = \prod_{\Lambda}(p\delta_1 + (1-p)\delta_0)$ be the Bernoulli product measure on Λ. The next two statements are proved in [12] (chapter IV) via the dual-countours methods

$$\lim_{p \to 1} \hat{\mathbb{P}}_p(|C_{(2,0)}| = \infty) = 1, \quad (1.3.4)$$

and for every $\beta \in (0,1)$

$$\lim_{p \to 1} \hat{\mathbb{P}}_p(\exists \ n \geq 1 : \hat{r}_n < \beta n) = 0. \quad (1.3.5)$$

The next result follows also by the dual-contours methods of Durrett; for details see [6].

Lemma 1.2. *For every t there exist $0 < \epsilon_t < 1$ and $0 < p_0 < 1$ such that for all $p_0 < p < 1$:*

$$\inf_{\substack{x \in [1,M],\ x \text{ even};\\ y \in [1,M],\ y + M^2 \text{ even}}} \hat{\mathbb{P}}_p((x,0) \rightsquigarrow (y,t) \text{ inside } [1,M]) \geq \epsilon_1,$$

for M large enough and $n \in [tM^2, 2tM^2]$.

The following lemma is a consequence of Theorem 0.0 in [13] and allows us to extend Lemma 1.2 for the *k-dependent* percolation system with closure close to 0.

Lemma 1.3. *Having fixed $k \in \mathbb{N}$ and $0 < p < 1$, there exists $\delta > 0$ such that if $(\Omega, \mathcal{F}, \mathbb{P})$ is a k-dependent oriented percolation system with closure below δ, then $\mathbb{P}$ stochastically dominates $\hat{\mathbb{P}}_p$.*

As a consequence of Lemma 1.2, we obtain the next corollary.

Corollary 1.1. *For any $t > 0$ and $0 < \epsilon < 1$ there exists a constant $c > 0$ such that if on the interval $[0,m]$ both A and C are subsets of $2\mathbb{Z}$ that intersect every interval of length $\sqrt{m}$, then for every 1-dependent oriented percolation closure smaller than ϵ we have that*

$$\mathbb{P}(\Psi_n^C = 0 \text{ on } A) \leq e^{-c\sqrt{m}},$$

for every $n \in [tm, 2tm]$ and sufficiently large m.

Proof. Let x_i and y_i be a pair of numbers such that $x_i \in C$, $y_i \in A$ and $x_i, y_i \in [(i-1)\sqrt{m}, i\sqrt{m})$ for $1 \leq i \leq \sqrt{m}$. Then

$$\begin{aligned}\mathbb{P}(\Psi_t^C = 0 \text{ on } A) &\leq \mathbb{P}\Big(\bigcap_{1 \leq i \leq \sqrt{m}} \{(x_i,0) \not\rightsquigarrow (y_i,t) \text{ inside } [(i-1)\sqrt{m}, i\sqrt{m})\}\Big)\\ &\leq \epsilon_t^{\sqrt{m}},\end{aligned}$$

where the last inequality is a consequence of Lemma 1.3 and Lemma 1.2. □

1.4 Mountford-Sweet renormalization

The contact process in dimension 1 and range $R = 1$ has a characteristic that is very useful to the study of this process. In this case, infection paths only make jumps of size one, therefore two paths cannot cross without intersecting. We refer to this as the paths crossing property. The contact process in dimension 1 and range $R > 1$ does not have this nature. A *k-dependent* percolation system Ψ with small closure is introduced in [16] and was used in [16] and [1] to avoid this problem. In Section 2.1, we present another use of this percolation system. Now we define Ψ.

Let $\hat{N}$ and $\hat{K}$ be to positive integers. Given $m \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$ such that $m+n$ is even, we define the following sets

$$\begin{aligned}
\mathcal{I}_m^{\hat{N}} &= \left(\frac{m\hat{N}}{2} - \frac{\hat{N}}{2}, \frac{m\hat{N}}{2} + \frac{\hat{N}}{2}\right] \cap \mathbb{Z}, \\
I_{(m,n)}^{\hat{N},\hat{K}} &= \mathcal{I}_m^{\hat{N}} \times \{\hat{K}\hat{N}n\}, \\
J_{(m,n)}^{\hat{N},\hat{K}} &= \left(\frac{m\hat{N}}{2} - R, \frac{m\hat{N}}{2} + R\right) \times [\hat{K}\hat{N}n, \hat{K}\hat{N}(n+1)].
\end{aligned}$$

We call the set

$$I_{(m,n)}^{\hat{N},\hat{K}} \cup J_{(m,n)}^{\hat{N},\hat{K}} \cup I_{(m,n+1)}^{\hat{N},\hat{K}}$$

the renormalized box corresponding to (m, n), or just the box (m, n).

For a realization $\mathcal{H}$ of the Harris construction, we construct a map $\Psi(\mathcal{H}) : \Lambda \to \{0, 1\}$ as follows: we set $\Psi(\mathcal{H})(m, n) = 1$ if the four conditions below are satisfied

$$\begin{gathered}
\text{For each interval } I \subset \mathcal{I}_{m-1}^{\hat{N}} \cup \mathcal{I}_{m+1}^{\hat{N}} \text{ of length } \sqrt{\hat{N}}, \\
\text{it holds } I \cap \xi_{\hat{N}(n+1)}^{\mathbb{1}} \neq \emptyset;
\end{gathered} \tag{1.4.1}$$

$$\begin{gathered}
\text{If } x \in \mathcal{I}_{m-1}^{\hat{N}} \cup \mathcal{I}_{m+1}^{\hat{N}} \text{ and } \xi_{\hat{K}\hat{N}n}^{\mathbb{1}} \times \{\hat{K}\hat{N}n\} \to (x, \hat{K}\hat{N}(n+1)), \\
\text{then } (\xi_{\hat{K}\hat{N}n}^{\mathbb{1}} \times \{\hat{K}\hat{N}n\}) \cap I_{(m,n)}^{\hat{N},\hat{K}} \to (x, \hat{K}\hat{N}(n+1));
\end{gathered} \tag{1.4.2}$$

$$\begin{gathered}
\text{If } x \in J_{(m,n)}^{\hat{N},\hat{K}} \text{ and } \xi_{\hat{K}\hat{N}n}^{\mathbb{1}} \times \{\hat{K}\hat{N}n\} \to (x, s), \\
\text{then } (\xi_{\hat{K}\hat{N}n}^{\mathbb{1}} \times \{\hat{K}\hat{N}n\}) \cap I_{(m,n)}^{\hat{N},\hat{K}} \to (x, s);
\end{gathered} \tag{1.4.3}$$

$$\left\{ \begin{array}{l} x \in \mathbb{Z} : \exists s, t, \hat{K}\hat{N}n \leq s < t \leq \hat{K}\hat{N}(n+1), \\ y \in \mathcal{I}_{m-1}^{\hat{N}} \cup \mathcal{I}_{m+1}^{\hat{N}} \text{ such that } (x,s) \to (y,t) \end{array} \right\} \subset \left[\frac{m\hat{N}}{2} - 2\alpha\hat{K}\hat{N}, \frac{m\hat{N}}{2} + 2\alpha\hat{K}\hat{N} \right]. \tag{1.4.4}$$

Otherwise, we set $\Psi(\mathcal{H})(m,n) = 0$.

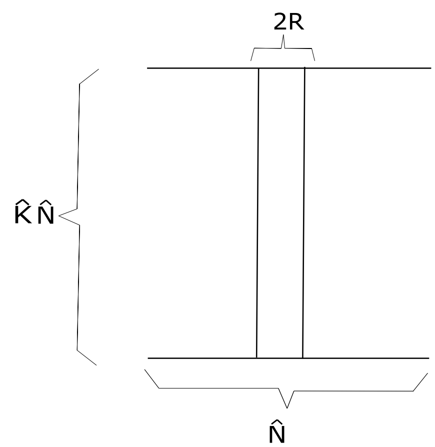

Figure 1.2: Mountford-Sweet renormalized site.

Several remarks are in order. First, equation (1.4.1) implies that there are many sites on the base of the boxes $(m-1,n)$ and $(m+1,n)$ which are connected in the Harris construction with $\mathbb{Z} \times \{0\}$. Second, equation (1.4.2) yields that if a site at the top of the box (m,n) is connected in the Harris construction with $\mathbb{Z} \times \{0\}$, then it is connected with the base of the box (m,n). Third, equation (1.4.3) guarantees that if a site in the rectangle $J_{(m,n)}^{\hat{N},\hat{K}}$ is connected with $\mathbb{Z} \times \{0\}$, then it is connected with the base of the box (m,n). Finally, equation (1.4.4) implies that every path connecting a site in the box (m,n) with

$\mathbb{Z} \times \{0\}$ is inside the rectangle

$$\left[\frac{m\hat{N}}{2} - 2\alpha\hat{K}\hat{N}, \frac{m\hat{N}}{2} + 2\alpha\hat{K}\hat{N}\right] \times [\hat{K}\hat{N}n, \hat{K}\hat{N}(n+1)].$$

The above rectangle is called the envelope of the box (m, n).

Additionally, we observe that the constant α in equation (1.4.4) is as in (1.1.9).

The following proposition, which was proved in [16], shows that we can construct Ψ with sufficiently small closure.

Proposition 1.1. *There are k and $\hat{K}$ with the property that for any $\delta > 0$ there is $\hat{N}_0$ such that Ψ is a k-dependent percolation system with closure under δ for all $\hat{N} > \hat{N}_0$.*

Throughout this work we fix

* k and $\hat{K}$ as in Proposition 1.1;
* p_0 as in Lemma 1.2;
* $\delta = \delta(k, p_0)$ as in Lemma 1.3;
* $\hat{N}_0 = \hat{N}_0(\delta, k, \hat{K})$ as in Proposition 1.1.
* $\hat{N} > \hat{N}_0$.

Under the above conditions, we have that Ψ is a *k-dependent* percolation system with closure under δ and with law stochastically larger than $\hat{\mathbb{P}}_{p_0}$.

1.5 Bezuidenhout-Grimmett renormalization

We dedicate this section to recall the Bezuidenhout-Grimmett renormalization introduced in [3]. In Chapter 3, we present an extension of the Mountford-Sweet renormalization for the layer $[-L, L] \times \mathbb{Z}$, for L fixed but arbitrary. For this extension we need the construction presented in [3] for $d = 2$. The idea presented in [3] is, roughly speaking, the construction of a renormalization in which a renormalized space-time block will be open if there exists a seed (finite region fully occupied) on the first level of the block connected inside the block to a translation of this seed in the last level of the block.

The next proposition is a very similar version of Lemma (19) in [3]. The main difference is that the version we present considers the seed as a rectangle with the same width than the layer. In the future, this specific detail helps to create "barriers" for the contact process in the layer. The concept of barriers is introduced in Section 3.2, intuitively the barriers stop the pass of particles of type 1 in the region favorable to type 2 and vice versa.

Let $\xi_{[-L,L]\times\mathbb{Z}}(t)$ be the contact process restricted to the layer $[-L,L]\times\mathbb{Z}$. We will work with the supercritical case in the layer. That is, we consider the infection parameter λ larger than λ_L, where

$$\lambda_L = \inf\{\lambda : \mathbb{P}_\lambda(\xi^0_{[-L,L]\times\mathbb{Z}}(t) \neq \emptyset\, \forall t) > 0\}. \tag{1.5.1}$$

Proposition 1.2. *Let $\lambda > \lambda_L$. Given $\delta > 0$, there exist integers r, K and T such that*

$$\mathbb{P}_\lambda\left(\begin{array}{c} \exists\, y' \in [9K, 13K] \text{ and } s \in [22T, 24T] \text{ such} \\ \text{that } ([-L,L]\times([-r,r]+y)\times\{t\} \to (z,s) \\ \forall z \in (0,y') + [-L,L]\times[-r,r] \text{ inside the region } \mathcal{R}^+ \end{array}\right) > 1-\delta \tag{1.5.2}$$

for all $y \in [-2K, 2K]$ and $t \in [0, 2T]$, where $\mathcal{R}^+$ is defined as follows

$$\mathcal{R}^+ = \left\{(x,y,t) : x \in [-L,L], t \in [0,24T], y \in \left[-5K + \frac{Kt}{2T}, 5K + \frac{Kt}{2T}\right]\right\}. \tag{1.5.3}$$

In order to define the 1-dependent percolation system, which is the main question of this section, we need more notation: denote $\mathcal{R}^-$ the reflection with respect to the axis t of $\mathcal{R}^+$ and for $(m,n) \in \Lambda$, denote $\mathcal{R}^\pm_{m,n} = \mathcal{R}^\pm + (0, m11K, 22nT)$. We also set

$$\mathcal{R} = \underset{(m,n)\in\Lambda}{\cup} (\mathcal{R}^+_{m,n} \cup \mathcal{R}^-_{m,n}).$$

For r, K, and T as in Proposition (1.2) we define a percolation system $\Phi \in \{0,1\}^\Lambda$ as follows: we set $\Phi(0,0) = 1$, for $n > 0$ we set $\Phi(m,n) = 1$ if the following two restrictions are satisfied:

i) There exist $y \in [-2K + 11mK, 2K + m11K]$ and $s \in [22Tn, 24Tn]$, such that $[-L,L]\times[-r+y, r+y]\times\{s\}$ is fully occupied;

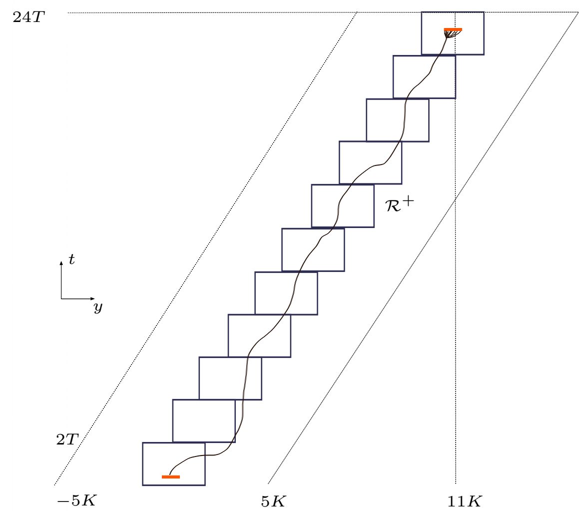

Figure 1.3: Representation of the event inside the probability in (1.5.2).

$$\text{ii)}\quad \begin{array}{c} \exists\,(x',y') \in [-L,L]\times([-9K+11mK,-13K+11mK]\cup[9K+11mK,13K+11mK]) \\ \text{and } s\in[22T+22nT,24T+22nT] \text{ such that, } [-L,L]\times[-r+y,r+y] \\ \times\{s\}\to(z,s) \quad \text{inside } \mathcal{R}^+_{m,n}\cup\mathcal{R}^-_{m,n}\forall z\in[-L,L]\times[-r+y',r+y'], \end{array}$$

we set $\Phi(m,n)=0$ otherwise. Restriction (ii) above says that there exists a translation of $[-L,L]\times[-r+y,r+y]\times\{s\}$ in the top of $\mathcal{R}^+_{m,n}\cup\mathcal{R}^-_{m,n}$ such that every point in this translation is connected inside $\mathcal{R}$ to $[-L,L]\times[-r+y,r+y]\times\{s\}$.

By Proposition 1.2 we have that Φ is a 1-dependent oriented percolation system with closure under δ.

Chapter 2

Metastability for the one-dimensional contact process with two types of particles and priority

During this chapter we are dealing with the supercritical contact process in dimension 1 with range $R \geq 1$. We study the metastability phenomenon for the contact process with two types of particles and priority. The first result on this matter is the convergence of $\tau_N^{\mathbf{1},\mathbf{2}}$, properly rescaled, to an exponential distribution (recall the definition of $\tau_N^{\mathbf{1},\mathbf{2}}$ in Section 1.2). More precisely, we prove that

Theorem 2.1. *For $d = 1$ and $R > 1$, let β_N be such that $\mathbb{P}(\tau_N^{\mathbf{1},\mathbf{2}} \geq \beta_N) = e^{-1}$, then*

$$\lim_{N\to\infty} \frac{\tau_N^{\mathbf{1},\mathbf{2}}}{\beta_N} = E \text{ in distribution,}$$

where E has exponential distribution with rate 1.

An important tool to obtain Theorem 2.1 is the concept of barriers introduced in Section 2.1. We will say that a point $(x, 0)$ in $\mathbb{Z} \times \mathbb{R}^+$ is a barrier in $[1, N] \times \{0\}$ if every open site in the interval at a fixed time, polynomial on N, is connected in the Harris graph to $(x, 0)$. Using Mountford-Sweet renormalization we state that with positive probability a point in

$[1, N] \times \{0\}$ is a barrier. Another ingredient for the proof of Theorem 2.1 is the property of regeneration of the contact process with range $R > 1$. This property is introduced in [14] in a more difficult situation, the contact process in $\mathbb{Z}^d$. In this paper is proved that the property of regeneration together with the attractiveness of the contact process implies that the normalized time of extinction in a box converge to an exponential distribution. In Section 2.3.1 we put together the regeneration for the contact process and the concept of barriers to obtain regeneration for the contact process with two types of particles and priority. Once we have regeneration for the process we are interested, Theorem 2.1 follows easy, although a little different than for the contact process, because in our case we do not have the property of attractiveness.

The other important result of this chapter is the following

Theorem 2.2. *For $d = 1$ and $R > 1$, there exists a constant $\gamma > 0$ depending only on the rate of infection λ and the range R such that*

$$\lim_{N\to\infty} \frac{1}{N} \log \tau_N^{\mathbf{1,2}} = \gamma \textit{ in probability .}$$

In Section 2.2 we also discuss the fact that the time of extinction for the contact process in the interval $[1, N]$ is logarithm equivalent to $e^{\gamma N}$, with γ as in Theorem 2.2. Theorem 2.2 implies that the time when one of the families became extinct in an interval of length $2N$ is logarithmically equivalent to $e^{\gamma N}$. Then after $\tau_N^{\mathbf{1,2}}$ the surviving family is alive in an interval of length $2N$ for an exponential time.

2.1 Barriers in finite volume

In this section, we introduce an object that we call *N-barrier*. To motivate the construction of this object, we observe the following property of the contact process with $R = 1$: Fix x in $[1, N]$, $D > 0$ and $t \in [DN^2, 2DN^2]$. Observe that by the path crossing property, in the event

$$\{(x, 0) \to (1, t) \text{ inside } [1, N]; (x, 0) \to (N, t) \text{ inside } [1, N]\},$$

we have that $\xi_N^x(t) = (\xi^{\mathbb{1}}(t) \cap [1, N])$. Corollary 1 in [16] establishes that for any $D > 0$, there is a constant δ such that

$$\mathbb{P}((z, 0) \to (y, t) \text{ inside } [1, N]) \geq \delta,$$

given any $z, y \in [1, N]$, $t \in [DN^2, 2DN^2]$ and sufficiently large N. By the FKG-inequality we have

$$\mathbb{P}((x, 0) \to (1, t) \text{ inside } [1, N]; (x, 0) \to (N, t) \text{ inside } [1, N]) \geq \delta^2,$$

which implies that there exists $\hat{\eta} = \delta^2$ such that

$$\inf_{x \in [1,N]} \mathbb{P}(\xi_N^x(t) = (\xi^{\mathbb{1}}(t) \cap [1, N])) > \hat{\eta} > 0. \tag{2.1.1}$$

The strong use of the path crossing property to obtain (2.1.1) restricts this argument for the case $R = 1$. We want to extend (2.1.1), or a similar equation, for the contact process with range $R > 1$. To this aim, we introduce the definition of an *N-barrier*, which is similar to the notion called descendancy barrier introduced in [1]. The main difference between these two concepts is that the *N-barrier* is defined in an interval with length depending on N, while the descendancy barrier is defined in the whole line. In this section, we establish some properties of *N-barriers* and follows closely Section 2.2 of [1].

Let us denote $M = M(N) = \lfloor \frac{2(N - 2\alpha \hat{K} \hat{N})}{\hat{N}} \rfloor$, this is the largest m such that the envelope of $(m, 0)$ is a subset of $[1, N] \times [0, \infty)$ and we set $S = S(N) = \hat{K}\hat{N}M^2 + 2$. Now, we are ready to introduce the definition of an *N-barrier*.

Definition 2.1. *(a) For $x \in [1, N]$ we say $(x, 0)$ is an N-barrier if for all $y \in [1, N]$ such that $\mathbb{Z} \times \{0\} \to (y, S)$ then $(x, 0) \to (y, S)$ inside $[1, N]$.*

(b) For $x \in [-N + 1, 0]$ we say $(x, 0)$ is an N-barrier if for all $y \in [-N + 1, 0]$ such that $\mathbb{Z} \times \{0\} \to (y, S)$ then $(x, 0) \to (y, S)$ inside $[-N + 1, 0]$.

*(c) For $x \in [-N+1, N]$ we say that a point (x, t) is an N-*barrier *if $(x, 0)$ is an N-barrier in $\Theta_{(0,t)}(\mathcal{H})$.*

We need to introduce some notation that will be used in the next result. Consider the following partition of the interval $[1, N]$

$$[1, N] = A_1 \cup A_2 \cup A_3,$$

where

$$A_1 = \left[1, \frac{\imath \hat{N}}{2} - \frac{\hat{N}}{2}\right), \quad A_2 = \left[\frac{\imath \hat{N}}{2} - \frac{\hat{N}}{2}, \frac{M\hat{N}}{2} + \frac{\hat{N}}{2}\right],$$
$$A_3 = \left(\frac{M\hat{N}}{2} + \frac{\hat{N}}{2}, N\right], \tag{2.1.2}$$

and

$$\imath = \imath(N) = \begin{cases} \lfloor 4\alpha \hat{K}\hat{N} \rfloor & \text{if } M^2 + \lfloor 4\alpha \hat{K}\hat{N} \rfloor \text{ is even,} \\ \lfloor 4\alpha \hat{K}\hat{N} \rfloor + 1 & \text{if } M^2 + \lfloor 4\alpha \hat{K}\hat{N} \rfloor \text{ is odd.} \end{cases} \tag{2.1.3}$$

Let $x \in A_2$, in the next result we prove that in the intersection of the following 5 events, E_i $i = 1, \ldots, 5$, $(x, 0)$ is an *N-barrier*. Take j such that $x \in \mathcal{I}_j^{\hat{N}}$, we define

$E_1=$ If y and z in $\mathcal{I}_{j-1}^{\hat{N}} \cup \mathcal{I}_j^{\hat{N}} \cup \mathcal{I}_{j+1}^{\hat{N}}$, $y \neq z$ and $(y,0) \to (z,t)$ with $t \leq 1$, then $(x,0) \to (z,t)$;

$E_2=$ $\{P^{\{x\}} \cap [0,1] = \emptyset\}$;

$E_3=$ $\{\forall z \in \mathcal{I}_{j-1}^{\hat{N}} \cup \mathcal{I}_j^{\hat{N}} \cup \mathcal{I}_{j+1}^{\hat{N}},\ (x,0) \to (z,1)\}$;

$E_4=$ $\{\Psi(\Theta_{(0,1)}(\mathcal{H})) \in \Gamma_M(j)\}$.

$E_5=$

$$\bigcap_{x,y \in A_1 \cup A_3} \{P^{x \to y} \cap (S-1, S] = \emptyset;\ \ P^{y \to x} \cap (S-1, S] = \emptyset;\ \ P^{x} \cap (S-1, S] \neq \emptyset\};$$

where Γ_M is defined as follows

$$\Gamma_M(j) = \left\{(j,0) \rightsquigarrow (\imath, M^2) \text{ and } (j,0) \rightsquigarrow (M, M^2) \text{ inside } \Lambda \cap ([\imath, M] \times [0, M^2])\right\}.$$

Figure 2.1 can help you to visualize the event $\cap_{i=1}^{5} E_i$. The following proposition states that for N large enough the probability of a (centrally located) point $(x,0)$ to be an *N-barrier* is uniformly bounded away from zero.

Proposition 2.1. *There exists $\hat{\eta} = \hat{\eta}(\lambda) > 0$ such that for all N large enough*

$$\mathbb{P}^x((x,0) \textit{ is an } N\textit{-barrier}) > \hat{\eta}, \tag{2.1.4}$$

for any $x \in \left[\frac{-M\hat{N}}{2} - \frac{\hat{N}}{2}, \frac{-\imath\hat{N}}{2} + \frac{\hat{N}}{2}\right] \cup \left[\frac{\imath\hat{N}}{2} - \frac{\hat{N}}{2}, \frac{M\hat{N}}{2} + \frac{\hat{N}}{2}\right]$.

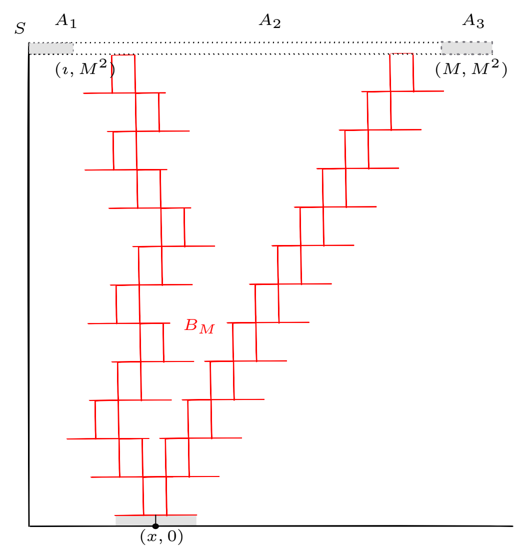

Figure 2.1: Event $E_1 \cap E_2 \cap E_3$ guarantees that every point in $\mathcal{I}_{j-1}^{\hat{N}} \cup \mathcal{I}_j^{\hat{N}} \cup \mathcal{I}_{j+1}^{\hat{N}}$ is connected with $(x, 0)$. Event E_4 implies that there are two renormalized paths connecting the box $(j, 0)$ with the boxes $(\imath, M^2)$ and (M, M^2). These renormalized paths are represented in the figure with the connected structure B_M (in red). Finally, the event E_5 ensures that there are no particles in the regions $A_1 \times [S-1, S]$ and $A_3 \times [S-1, S]$ and this event is represented in the figure with two gray rectangles at the top.

Proof. Observe that for N large enough $\frac{3}{2\hat{N}}N \leq M \leq \frac{2}{\hat{N}}N$ and $\hat{K}\frac{9}{4\hat{N}}N^2 \leq S \leq \hat{K}\frac{4}{\hat{N}}N^2$. Then for $D = \hat{K}\frac{9}{4\hat{N}}$, $DN^2 \leq S \leq 2DN^2$ and (2.1.4) we have (2.1).

We now give a proof that works for all values of R. We concentrate in the case $x \in A_2$. For a configuration in E_4 there exist sequences $\{m_k\}_{0\leq k\leq M^2}$ and $\{\hat{m}_k\}_{0\leq k\leq M^2}$, subsets of $\{\imath, \ldots, M\}$, such that

$$m_0 = \hat{m}_0 = j, \quad m_{M^2} = \imath, \quad \hat{m}_{M^2} = M,$$

and

$$\begin{aligned}
\Psi(\Theta_{(0,1)}(\mathcal{H}))(m_k, k) = 1 \quad &\text{and} \quad |m_{k+1} - m_k| = 1 \ \forall \ k \ \in \{0, \ldots, M^2\}, \\
\Psi(\Theta_{(0,1)}(\mathcal{H}))(\hat{m}_k, k) = 1 \quad &\text{and} \quad |\hat{m}_{k+1} - \hat{m}_k| = 1 \ \forall \ k \ \in \{0, \ldots, M^2\}.
\end{aligned}$$

Denote

$$B_M = \bigcup_{0\leq k\leq M^2} I^{\hat{K},\hat{N}}_{(m_k,k)} \cup J^{\hat{K},\hat{N}}_{(m_k,k)} \cup I^{\hat{K},\hat{N}}_{(\hat{m}_k,k)} \cup J^{\hat{K},\hat{N}}_{(\hat{m}_k,k)}.$$

From items (1.4.2) and (1.4.3) in the definition of Mountford-Sweet renormalization it follows that in the trajectory of the contact process $t \mapsto \xi(t)(\Theta_{(0,1)}(\mathcal{H}))$ every occupied site in B_M descends from $I^{\hat{K},\hat{N}}_{(j,0)}$. By our choice of N and $\imath$, we have that

$$\left[\frac{m_k\hat{N}}{2} - 2\alpha\hat{K}\hat{N}, \frac{m_k\hat{N}}{2} + 2\alpha\hat{K}\hat{N}\right] \subset [1, N] \text{ for all } k.$$

Therefore, the envelopes of the renormalized sites (m_k, k) and $(\hat{m}_k, k)$ are subsets of $[1, N] \times [0, \infty)$ for all k. Using (1.4.4) of the Mountford-Sweet renormalization, we have that every occupied site in B_M is connected to $I^{\hat{K},\hat{N}}_{(j,0)}$ by a path entirely contained in $[1, N] \times [0, \infty)$.

B_M is a connected union of M^2 segments of length $\hat{N}$ with rectangles of width $2R$ and height $\hat{K}\hat{N}$. In $\Theta_{(0,1)}(\mathcal{H})$, at time $\hat{K}\hat{N}M^2 = S-2$ every occupied site in A_2 is connected to $\mathbb{Z}\times\{0\}$ by a path that intersects the structure B_M and remains in $[1, N]\times[0, \infty)$ afterwards. Since every point in B_M is connected with $I^{\hat{K},\hat{N}}_{(j,0)}$ inside $[1, N]$, we also can connect every point in $A_2 \times \{S - 2\}$ with $I^{\hat{K},\hat{N}}_{(j,0)}$ inside $[1, N] \times [0, \infty)$, in the construction $\Theta_{(0,1)}(\mathcal{H})$.

The intersection of the events E_1, E_2 and E_3 implies that every point in $\mathcal{I}^{\hat{N}}_j \times \{1\}$ is connected with $(x, 0)$, in $\mathcal{H}$. Since $\mathcal{I}^{\hat{N}}_j \times \{1\}$ is the basis of $I^{\hat{K},\hat{N}}_{(j,0)}$, we have that for all $y \in A_2$ such that $\mathbb{Z} \times \{0\} \to (y, S - 1)$, $(x, 0) \to (y, S - 1)$ inside $[1, N]$.

Finally, for any realization in E_5 there is no mark of death in the regions $A_1 \times [S-1, S]$ and $A_3 \times [S-1, S]$, and also there is no infection mark going out or coming in these regions. In particular, for any initial configuration at time S there is no particle alive in $A_1 \cup A_3$, and during the interval of time $[S-1, S]$ there is no interaction with any exterior regions. Therefore, every occupied site at time S is connected with $A_2 \times \{S-1\}$ inside A_2. Then we can conclude that every occupied site in $[1, N]$ at time S is connected with $(x, 0)$ inside $[1, N]$ and this is the definition of N-barrier.

Now we proceed to prove that the probability of $\cap_{i=1}^5 E_i$ is positive. It is trivial that we can take $\tilde{p} > 0$ independent of N so that $P(E_i) \geq \tilde{p}$ for $i = 1, 2, 3$. Since the event $\Psi(\Theta_{(0,1)}(\mathcal{H})) \in \Gamma_M(j)$ depends on the Harris construction restricted to $\mathbb{Z} \times [1, S-1)$, it is independent of all the marks in $\mathbb{Z} \times [0, 1)$. Let us prove that the event $\Psi(\Theta_{(0,1)}(\mathcal{H})) \in \Gamma_M(j)$ has positive probability.

Using the FKG-inequality and Lemma 1.2 we have that for all M large enough

$$\hat{\mathbb{P}}_{p_0}((j, 0) \rightsquigarrow (\imath, M^2) \text{ and } (j, 0) \rightsquigarrow (M-1, M^2) \text{ inside } [\imath, M]) > {\epsilon_1}^2.$$

By our selection of Ψ, the law of Ψ is stochastically larger than $\hat{\mathbb{P}}_{p_0}$, then

$$\mathbb{P}(\Psi(\Theta_{(0,1)}(\mathcal{H})) \in \Gamma_M(j)) > \epsilon_1^2.$$

On the other hand, the event E_5 depends on marks in the region $\mathbb{Z} \times [S-1, S]$. Note that this event has probability

$$\mathbb{P}(E_5) = \left(\frac{1 - e^{-(R\lambda+1)}}{R\lambda + 1} \right)^{\frac{\imath\hat{N}}{2} - \frac{\hat{N}}{2}} \left(\frac{1 - e^{-(R\lambda+1)}}{R\lambda + 1} \right)^{N - \frac{M\hat{N}}{2} - \frac{\hat{N}}{2}}. \tag{2.1.5}$$

Therefore, since $N - \frac{M\hat{N}}{2} - \frac{\hat{N}}{2} \leq 2\alpha\hat{K}\hat{N} - \frac{\hat{N}}{2}$, we conclude that $\mathbb{P}(E_5)$ has a positive lower bound, say β, that does not depend on N. Thus, by the Markov property, (2.1.4) holds for $\hat{\eta} = \tilde{p}\epsilon_1^2\beta$.

Finally, for $x \in \left[\frac{-M\hat{N}}{2} - \frac{\hat{N}}{2}, \frac{-\imath\hat{N}}{2} + \frac{\hat{N}}{2}\right]$ the proof is analogous by the symmetry of Harris graph. $\square$

2.2 Regeneration for the contact process

This section deals with the "regeneration" property of the contact process with range $R \geq 1$ restricted to the interval $[1, N]$ for large N. For any initial configuration, in a set of probability close to one, if the process survives until a certain time a_N then the infected sites are the same as if the process had started with full occupancy. In addition, a_N is negligeable compared with the extinction time. The notion of regeneration was introduced in [14] for the contact process in dimension $d \geq 2$.

Proposition 2.2. *There exist sequences a_N and b_N that satisfy*

(i) $\lim_{N\to\infty} \inf_{\xi_0 \in \{0,1\}^{[1,N]}} \mathbb{P}(\xi_N^{\mathbf{1}}(a_N) = \xi_N^{\xi_0}(a_N) \text{ or } T_N < a_N) = 1,$

(ii) $\frac{b_N}{a_N} \to \infty,$

(iii) $\lim_{N\to\infty} \mathbb{P}(T_N^{\mathbf{1}} < b_N) = 0.$

Remark 2.1. *In the sequel we take $a_N = (\hat{K}\hat{N}M^2 + 3)N$ and $b_N = e^{\frac{\gamma}{2}N}$.*

We restrict the proof of this proposition to the case $R > 1$. For the nearest neighbour contact process the idea is the same just replacing the use of the object *N-barrier* by the property mentioned in (2.1.1).

Proof of Proposition 2.2. We start by proving item (i). Fix N large enough as in Proposition 2.1 and remember the definition of $M = M(N)$, $S = S(N)$ before Definition 2.1, $\imath = \imath(N)$ in (2.1.3) and the interval A_2 in (2.1.2). For $i \in \mathbb{N}$, we define

$$B_i = \{\exists\, x \in A_2 : \xi_N^{\xi_0}(s_{i-1}+1)(x) = 1 \text{ and } (x, s_{i-1}+1) \text{ is } N\text{-}barrier\},$$

where $s_i = (S+1)i$ and $s_0 = 0$. Observe that by the Markov property

$$\mathbb{P}(\exists\, x \in A_2 \text{ such that } \xi_N^{\xi_0}(s_{i-1}+1)(x) = 1 | T_N^{\xi_0} > s_{i-1}) \geq \min_{y\in[1,N]} \mathbb{P}((y,0) \to A_2 \times \{1\}).$$

By the definition of A_2 and $\imath$, the left extreme of the interval A_2 is at most $(\lfloor 4\alpha\hat{K}\hat{N} \rfloor + 1)\hat{N}/2 - \hat{N}/2$ implies that we can choose $\tilde{\eta} > 0$ that does not depend on N such that

$$\min_{y\in[1,N]} \mathbb{P}((y,0) \to A_2 \times \{1\}) \geq \tilde{\eta}.$$

Using Markov property and Proposition 2.1 we obtain

$$\mathbb{P}(B_i | T_N^{\xi_0} > s_{i-1}) \geq \hat{\eta}\tilde{\eta}.$$

Note that B_i involves information between the times $s_{i-1}+1$ and s_i. Hence, setting $a_N = (S+1)N$ by the Markov property we have

$$\mathbb{P}(\bigcap_{i\leq N} B_i^c \cap \{T_N^{\xi_0} > a_N\}) \leq (1-\hat{\eta}\tilde{\eta})^N. \tag{2.2.1}$$

Item (i) will follow by (2.2.1) and the following inclusion

$$\bigcup_{i\leq N} B_i \cup \{T_N^{\xi_0} \leq a_N\} \subset \{\xi_N^{\mathbf{1}}(a_N) = \xi_N^{\xi_0}(a_N)\} \cup \{T_N^{\xi_0} \leq a_N\}.$$

To obtain this inclusion is enough to prove

$$\bigcup_{i\leq N} B_i \cap \{T_N^{\xi_0} > a_N\} \subset \{\xi_N^{\mathbf{1}}(a_N) = \xi_N^{\xi_0}(a_N)\}. \tag{2.2.2}$$

Fix a realization in the left member of (2.2.2) and take i such that $\xi_N^{\xi_0}(x) = 1$ and $(x, s_{i-1}+1)$ is an N-*barrier*, then we have by the definition of N-*barrier*

$$\forall\, y \in [1,N] \text{ such that } \mathbb{Z}\times\{s_{i-1}+1\} \to (y, s_i) \text{ then } (x, s_{i-1}+1) \to (y, s_i) \text{ inside } [1,N],$$

which establishes (2.2.2).

For the item (iii) we use the next result: there exists $\gamma \in (0,\infty)$ such that

$$\lim_{N\to\infty} \frac{1}{N} \log T_N^{\mathbf{1}} = \gamma \text{ in probability.} \tag{2.2.3}$$

We discuss this result in the following remark. By (2.2.3) we can take $b_N = e^{\frac{\gamma}{2}N}$ for item (iii). Item (ii) is immediate from the selection of a_N and b_N.

□

Remark 2.2. *For the nearest neighbor scenario it was shown in [8] that for any $\epsilon > 0$*

$$\lim_{N\to\infty} \mathbb{P}\left(\frac{1}{N}\log T_N^{\mathbf{1}} > \gamma + \epsilon\right) = 0, \tag{2.2.4}$$

and in [9] it was proved the other bound

$$\lim_{N\to\infty} \mathbb{P}\left(\frac{1}{N}\log T_N^{\mathbf{1}} < \gamma - \epsilon\right) = 0. \tag{2.2.5}$$

These results imply (2.2.3) *for* $R = 1$. *Both statements use the fact that there exists* $\hat{c} > 0$ *such that for all* $t \geq 0$

$$\mathbb{P}(t < T^{[1,N]} < \infty) \leq e^{-t\hat{c}}, \tag{2.2.6}$$

proved in [7]. Formula (2.2.6) *is obtained by the Peierls contour argument. When* $R > 1$, *we can still use the same argument to obtain* (2.2.6) *except that the renormalization used in the previous case is replaced by the Mountford-Sweet renormalization. All the other steps in the proof of* (2.2.4) *and* (2.2.5) *for the case nearest neighbor are also valid when* $R > 1$.

Having proven the regeneration property, we get, just as in Proposition 1.2 of [14], the following extension of the asymptotic exponentiality, valid for the case $R > 1$.

Corollary 2.1.

$$\lim_{N\to\infty} \frac{T_N^{\mathbf{1}}}{\mathbb{E}(T_N^{\mathbf{1}})} = E \textit{ in distribution},$$

where E *has exponential distribution with rate* 1.

2.3 Metastability for the contact process with two types of particles and priority

In this section, we prove Theorem 2.1. We start by proving a proposition which establishes that if the time of the first extinction is larger than a_{2N} (a_N as in Proposition 2.2), then there exists a time smaller than a_{2N} such that at least one particle of type 2 is in $[1, N]$, the region in which the particles 2 have priority.

Given $\zeta_0 \in \{0, 1, 2\}^{\mathbb{Z}}$, $k \geq 1$ and N, define the following stopping times

$$S_k^{\zeta_0} = \inf\{t > (k-1)a_{2N} : \exists\, y \in [1, N],\ \zeta_t^{\zeta_0, N}(y) = 2\}$$

and

$$\hat{S}_k^{\zeta_0} = \inf\{t > (k-1)a_{2N} : \exists\, x \in [-N+1, 0],\ \zeta_t^{\zeta_0, N}(x) = 1\}.$$

Note that, by the symmetry in Harris construction, $\hat{S}_k^{\zeta_0}$ and $S_k^{\zeta_0}$ have the same distribution. Hence, we state the following result for $S_k^{\zeta_0}$, but it is also valid for $\hat{S}_k^{\zeta_0}$.

Proposition 2.3. *Let $\mathcal{C} = \{\zeta_0 \in \{0,1,2\}^{[-N+1,N]} : \exists\ x,y\ \zeta_0(x) = 1, \zeta_0(y) = 2\}$. Then there exists c, $0 < c < 1$, such that*

$$\sup_{\zeta_0\in\mathcal{C}} \mathbb{P}(\tau_N^{\zeta_0} > Na_{2N}; \exists\, k\ \ 1 \le k \le N : S_k^{\zeta_0} > ka_{2N}) \le Nc^{2N}, \tag{2.3.1}$$

for all N large enough.

Proof of Proposition 2.3. By the Markov property for $k \ge 2$ we have that

$$\begin{aligned}\mathbb{P}(\tau_N^{\zeta_0} > Na_{2N}; S_k^{\zeta_0} > ka_{2N}) &\le \sum_{\hat{\zeta}\in\mathcal{C}} \mathbb{P}(\tau_N^{\hat{\zeta}} > a_{2N}; S_1^{\hat{\zeta}} > a_{2N})\mathbb{P}(\zeta_{(k-1)a_{2N}}^{\zeta_0,N} = \hat{\zeta}) \\ &\le \sup_{\zeta_0\in\mathcal{C}} \mathbb{P}(\tau_N^{\zeta_0} > a_{2N}, S_1^{\zeta_0} > a_{2N}).\end{aligned}$$

Thus, to obtain (2.3.1) it is enough to prove

$$\sup_{\zeta_0\in\mathcal{C}} \mathbb{P}(\tau_N^{\zeta_0} > a_{2N}; S_1^{\zeta_0} > a_{2N}) \le c^{2N}, \tag{2.3.2}$$

for some c, $0 < c < 1$.

Only during this proposition we abuse notation and denote ξ_{2N} by the contact process restricted to the interval $[-N+1, N]$. By the translation invariance in the law of Harris construction, both processes have the same distribution.

Take $\zeta_0 \in \mathcal{C}$ and set $B = B(\zeta_0) = \{x : \zeta_0(x) = 1\}$. We claim that

$$\{\tau_N^{\zeta_0} > a_{2N}; S_1^{\zeta_0} > a_{2N}; \xi_{2N}^B(a_{2N}) = \xi_{2N}^{\mathbf{1}}(a_{2N})\} = \emptyset. \tag{2.3.3}$$

This claim implies that

$$\{\tau_N^{\zeta_0} > a_{2N}; S_1^{\zeta_0} > a_{2N}\} \subset \{T_{2N}^B > a_{2N}; \xi_{2N}^B(a_{2N}) \ne \xi_{2N}^{\mathbf{1}}(a_{2N})\}, \tag{2.3.4}$$

and hence (2.3.2) is a consequence of formulas (2.2.1) and (2.2.2).

To prove (2.3.3), it is enough to show that every realization in $\{S_1^{\zeta_0} > a_{2N}; \xi_{2N}^B(a_{2N}) = \xi_{2N}^{\mathbf{1}}(a_{2N})\}$ is in $\{\tau_N^{\zeta_0} \le a_{2N}\}$. Take $x \in \xi_{2N}^{\mathbf{1}}(a_{2N})$ and let γ be a path connecting $B \times \{0\}$

with (x, a_{2N}). For γ we define s^* by

$$s^* = \inf\{t : 0 < t \leq a_{2N}, \zeta_{2N}^{\zeta_0}(t)(\gamma(t)) = 2\},$$

with the usual convention that $\inf\{\emptyset\} = \infty$.

Suppose that $s^* < \infty$. Since $S_1^{\zeta_0} > a_{2N}$ and $\zeta_{s^*}^{\zeta_0,2N}(\gamma(s^*)) = 2$, we conclude that $\gamma(s^*) \in [-N+1, 0]$. However, by the definition of s^*, $\zeta_t^{\zeta_0,2N}(\gamma(t)) = 1$ for all $t < s^*$, which implies that γ restricted to $[0, s^*]$ is a path of particles 1 that infects the site $\gamma(s^*)$ at time s^*. Since the particles of type 1 have priority in $[-N+1, 0]$, we get $\zeta_{s^*}^{\zeta_0,2N}(\gamma(s^*)) = 1$. This is a contradiction and we conclude that $s^* = \infty$, which means $\zeta_t^{\zeta_0,2N}(\gamma(t)) = 1$ for all $0 \leq t \leq a_{2N}$. Therefore, $\zeta_{a_{2N}}^{\zeta_0,2N}(x) = 1$ for all $x \in \xi_{2N}^{\mathbf{1}}(a_{2N})$.

□

2.3.1 Regeneration for the contact process with two types of particles and priority

We are ready to state the regeneration phenomenon for the process $\{\zeta_t^{\mathbf{1},\mathbf{2},N}\}$. The main idea is to prove that if the two families of particles survive for a suitable time (polynomial in N), outside an event with exponentially small probability we can find two *barriers* one in $[-N+1, 0]$ and the other in $[1, N]$, each of them infected by the type of particle that has priority in the respective region. Basically, we combine the idea of the proof of Proposition 2.2 with the result in Proposition 2.3 to obtain the following:

Proposition 2.4. *There are sequences c_N and d_N that satisfy*

(i) $\lim_{N\to\infty} \inf_{\zeta_0 \in \mathcal{C}} \mathbb{P}(\zeta_{c_N}^{\mathbf{1},\mathbf{2},N} = \zeta_{c_N}^{\zeta_0,N} \text{ or } \tau_N^{\zeta_0} < c_N) = 1;$

(ii) $\frac{d_N}{c_N} \to \infty;$

iii) $\lim_{N\to\infty} \mathbb{P}(\tau_N^{\mathbf{1},\mathbf{2}} < d_N) = 0,$

where $\mathcal{C}$ has been defined in Proposition 2.3.

Proof. Observe that $\tau_N^{\mathbf{1},\mathbf{2}}$ is stochastically larger than the minimum of two independent variables with the same law as $T_N^{\mathbf{1}}$. Hence, taking d_N as b_N of Proposition 2.2, item (*iii*) is immediate.

For the proof of item (i), we take N large enough as in Proposition 2.1 and remember the definitions of $M = M(N)$, $S = S(N)$ before Definition 2.1, $\imath = \imath(N)$ in (2.1.3) and the interval A_2 in (2.1.2). We also denote $A_4 = \left[\frac{-M\hat{N}}{2} - \frac{\hat{N}}{2}, \frac{-\imath\hat{N}}{2} + \frac{\hat{N}}{2}\right]$. For $k \in \{1, \dots, N\}$, let $s_k = k2Na_{2N}$,

$$\Lambda^{\zeta_0}_{N,k} = \left\{ \begin{array}{l} \exists\, y_k \in A_2, z_k \in A_4 : \zeta^{\zeta_0,N}_{s_k}(y_k) = 2, \zeta^{\zeta_0,N}_{s_k}(z_k) = 1, \\ \text{and } (y_k, s_k), (z_k, s_k), \text{ are } N\text{-}barriers \end{array} \right\}$$

and

$$\Lambda_{N,k} = \left\{ \begin{array}{l} \exists\, \hat{y}_k \in A_2, \hat{z}_k \in A_4 : \zeta^{\mathbf{1},\mathbf{2},N}_{s_k}(\hat{y}_k) = 2, \zeta^{\mathbf{1},\mathbf{2},N}_{s_k}(\hat{z}_k) = 1, \\ \text{and } (\hat{y}_k, s_k), (\hat{z}_k, s_k) \text{ are } N\text{-}barriers \end{array} \right\}.$$

To obtain item (i), we prove the following inclusion

$$\begin{aligned} &\{\tau^{\zeta_0}_N > 2N^2a_{2N}; \tau^{\mathbf{1},\mathbf{2}}_N > 2N^2a_{2N}\} \cap \bigcup_{k=1}^{N} \Lambda^{\zeta_0}_{N,k} \cap \Lambda_{N,k} \subset \\ &\{\tau^{\zeta_0}_N > 2N^2a_{2N}; \tau^{\mathbf{1},\mathbf{2}}_N > 2N^2a_{2N}; \zeta^{\zeta_0,N}_{2N^2a_{2N}} = \zeta^{\mathbf{1},\mathbf{2},N}_{2N^2a_{2N}}\}. \end{aligned} \tag{2.3.5}$$

Fix a realization in $\{\tau^{\zeta_0}_N > 2N^2a_{2N}; \tau^{\mathbf{1},\mathbf{2}}_N > 2N^2a_{2N}\} \cap \Lambda^{\zeta_0}_{N,k} \cap \Lambda_{N,k}$. By the definition of N-*barrier*, the points (y_k, s_k), $(\hat{y}_k, s_k)$, (z_k, s_k) and $(\hat{z}_k, s_k)$ satisfy:

> If $y \in [1, N]$ and $\mathbb{Z} \times \{s_k\} \to (y, s_k + S)$, then $(y_k, s_k) \to (y, s_k + S)$ inside $[1, N]$ and $(\hat{y}_k, s_k) \to (y, s_k + S)$ inside $[1, N]$.
>
> If $z \in [-N + 1, 0]$ and $\mathbb{Z} \times \{s_k\} \to (z, s_k + S)$, then $(z_k, s_k) \to (z, s_k + S)$ inside $[1, N]$ and $(\hat{z}_k, s_k) \to (z, s_k + S)$ inside $[-N + 1, 0]$.

From the argument above for this realization, we conclude that

$$\zeta^{\zeta_0,N}_{s_k+S} = \zeta^{\mathbf{1},\mathbf{2},N}_{s_k+S} = 2\xi^{\mathbf{1}}_{[-N+1,N]}(s_k + S) \text{ in } [1, N]$$

and

$$\zeta^{\zeta_0,N}_{s_k+S} = \zeta^{\mathbf{1},\mathbf{2},N}_{s_k+S} = \xi^{\mathbf{1}}_{[-N+1,N]}(s_k + S) \text{ in } [-N + 1, 0],$$

consequently $\zeta^{\zeta_0,N}_{2N^2a_{2N}} = \zeta^{\mathbf{1},\mathbf{2},N}_{2N^2a_{2N}}$, which proves (2.3.5).

To finish the proof of item (i) we choose $c_N = 2N^2a_{2N}$ and show that there exists $0 < \nu < 1$ such that

$$\mathbb{P}\left(\{\tau_N^{\zeta_0} > 2N^2 a_{2N}; \tau_N^{\mathbf{1,2}} > 2N^2 a_{2N}\} \cap \bigcap_{k=1}^{N} (\Lambda_{N,k}^{\zeta_0} \cap \Lambda_{N,k})^c\right) \leq \nu^N \tag{2.3.6}$$

for all $\zeta_0 \in \mathcal{C}$. Observe that

$$\begin{aligned} &\mathbb{P}\left(\{\tau_N^{\zeta_0} > 2N^2 a_{2N}; \tau_N^{\mathbf{1,2}} > 2N^2 a_{2N}\} \cap \bigcap_{k=1}^{N} (\Lambda_{N,k}^{\zeta_0} \cap \Lambda_{N,k})^c\right) \leq \\ &4\sup_{\zeta_0 \in \mathcal{C}} \mathbb{P}\left(\{\tau_N^{\zeta_0} > 2N^2 a_{2N}\} \cap \bigcap_{k=1}^{N} \left\{ \begin{array}{l} \forall y \ \in A_2 \ \zeta_{s_k}^{\zeta_0,N}(y) \neq 2 \text{ or } (y, s_k) \\ \text{is not an } N\text{-}barrier \end{array} \right\}\right), \end{aligned} \tag{2.3.7}$$

which easily follows after opening the complement in the event on the left member and using the symmetry of the Harris construction. Now, by Proposition 2.3 we have

$$\begin{aligned} &\sup_{\zeta_0 \in \mathcal{C}} \mathbb{P}\left(\{\tau_N^{\zeta_0} > 2N^2 a_{2N}\} \cap \bigcap_{k=1}^{N} \left\{ \begin{array}{l} \forall y \ \in A_2 \ \zeta_{s_k}^{\zeta_0,N}(y) \neq 2 \text{ or } (y, s_k) \\ \text{is not an } N\text{-}barrier \end{array} \right\}\right) \leq \frac{N}{2} c^N + \\ &\sup_{\zeta_0 \in \mathcal{C}} \mathbb{P}\left(\{\tau_N^{\zeta_0} > 2N^2 a_{2N}\} \cap \bigcap_{k=1}^{N} \left\{ \begin{array}{l} S_{2kN-1}^{\zeta_0} \leq s_k - a_{2N} \ \forall y \ \in A_2 \ \zeta_{s_k}^{\zeta_0,N}(y) \neq 2 \\ \text{or } \zeta_{s_k}^{\zeta_0,N}(y) = 2 \text{ and } (y, s_k) \text{ is not an } N\text{-}barrier \end{array} \right\}\right). \end{aligned} \tag{2.3.8}$$

Then, it is enough to prove that the last term in the inequality above is exponentially small in N. To show this, we first observe that the last term in (2.3.8) is smaller than

$$\begin{aligned} &\sup_{\zeta_0 \in \mathcal{C}} \mathbb{P}\left(\{\tau_N^{\zeta_0} > 2N^2 a_{2N}\} \cap \bigcap_{k=1}^{N} \left\{ S_{2kN-1}^{\zeta_0} \leq s_k - a_{2N}; \ \forall y \ \in A_2 \ \zeta_{s_k}^{\zeta_0,N}(y) \neq 2 \right\}\right) \\ &+ \sup_{\zeta_0 \in \mathcal{C}} \mathbb{P}\left(\{\tau_N^{\zeta_0} > 2N^2 a_{2N}\} \cap \bigcap_{k=1}^{N} \left\{ \begin{array}{c} \exists \, y \ \in A_2 \ \zeta_{s_k}^{\zeta_0,N}(y) = 2 \text{ but} \\ (y, s_k) \text{ is not an } N\text{-}barrier \end{array} \right\}\right). \end{aligned} \tag{2.3.9}$$

Thus, we need only to obtain upper bounds for the members on the right side of (2.3.9). Let us start by the first term. Define

$$D_k = \{x : \zeta_{S_{2kN-1}^{\zeta_0}}^{\zeta_0,N}(x) = 2\} \cap [1, N]$$

and the event

$$C_k = \{\{D_k\} \times \{S_{2kN-1}^{\zeta_0}\} \nrightarrow A_2 \times \{S_{2kN-1}^{\zeta_0} + 2a_{2N}\} \text{ inside } [1, N]\}.$$

By the priority of particles of type 2 in $[1,N]$ we have that

$$\{\tau_N^{\zeta_0} > 2N^2 a_{2N}\} \cap \bigcap_{k=1}^{N} \left\{ S_{2kN-1}^{\zeta_0} \le s_k - a_{2N};\ \forall y\ \in A_2\ \zeta_{s_k}^{\zeta_0,N}(y) \neq 2 \right\}$$
$$\subset \{\tau_N^{\zeta_0} > 2N^2 a_{2N}\} \cap \bigcap_{k=1}^{N} \{S_{2kN-1}^{\zeta_0} \le s_k - a_{2N}; C_k\}.$$

In the following claim we prove that the conditional probabilities of the events C_k given $S_{2kN-1}^{\zeta} < s_k - a_{2N}$ are smaller than a positive number β.

Claim 2.1. *There exists $\beta > 0$ such that for all N large enough*

$$\mathbb{P}(C_k^c | S_{2kN-1}^{\zeta_0} < s_k - a_{2N}) > \beta.$$

Proof of Claim 2.1. Fix $\xi_0 \in \{0,1\}^{[1,N]}$ and $\xi_0 \neq \emptyset$, then we have

$$\begin{aligned} \mathbb{P}(\xi_N^{\xi_0}(2a_{2N}) \cap A_2 \neq \emptyset) &= \mathbb{P}(T_N^{\xi_0} > 2a_{2N}; \xi_N^{\xi_0}(2a_{2N}) \cap A_2 \neq \emptyset) \\ &= \mathbb{P}(T_N^{\xi_0} > 2a_{2N}) - \mathbb{P}(T_N^{\xi_0} > 2a_{2N}; \xi_N^{\xi_0}(2a_{2N}) \cap A_2 = \emptyset). \end{aligned} \tag{2.3.10}$$

Using a Peirels contour argument for the k-dependent system with small closure Ψ defined in Section 2.1, it is possible to prove that there exist $\beta > 0$ and a sequence f_N linear in N such that

$$\inf_{x \in [1,N]} \mathbb{P}(T_N^x \ge e^{f_N}) > 2\beta,$$

for N large enough. Since a_{2N} is of order N^3, the formula above implies that

$$\mathbb{P}(T_N^{\xi_0} > 2a_{2N}) \ge 2\beta. \tag{2.3.11}$$

Now, we prove that the last term in (2.3.10) goes to zero when N goes to infinity. Formulas (2.2.1) and (2.2.2) imply that there exists $0 < c < 1$ such that

$$\mathbb{P}(T_N^{\xi_0} > 2a_{2N}; \xi_N^{\xi_0}(2a_{2N}) \cap A_2 = \emptyset) \le c^N + \mathbb{P}(\xi_{2a_{2N}}^{\mathbf{1},N} \cap A_2 = \emptyset). \tag{2.3.12}$$

By the duality of the contact process we have

$$\mathbb{P}(\xi_N^{\mathbf{1}}(2a_{2N}) \cap A_2 \neq \emptyset) = \mathbb{P}(\xi_N^{A_2}(2a_{2N}) \neq \emptyset).$$

Observe that the length of A_2 is at least $l_N = N - 2\alpha\hat{K}\hat{N} - \left\lfloor 4\alpha\hat{K}\hat{N} \right\rfloor - 1$, then, we obtain

$$\mathbb{P}(\xi_N^{\mathbf{1}}(2a_{2N}) \cap A_2 = \emptyset) = \mathbb{P}(\xi_N^{A_2}(2a_{2N}) = \emptyset) \leq \mathbb{P}(\xi_{l_N}^{\mathbf{1}}(2a_{2N}) = \emptyset).$$

From item (iii) of Proposition 2.2 and the fact that l_N is linear in N, it follows that

$$\lim_{N\to\infty} \mathbb{P}(\xi_{l_N}^{\mathbf{1}}(2a_{2N}) = \emptyset) = \lim_{N\to\infty} \mathbb{P}(T_{l_N}^{\mathbf{1}} \leq 2a_N) = 0.$$

Thus, for N large enough we have

$$\mathbb{P}(T_N^{\xi_0} > 2a_{2N}; \xi_N^{\xi_0}(2a_{2N}) \cap A_2 = \emptyset) \leq \beta.$$

By (2.3.11) and (2.3.12) we obtain

$$\mathbb{P}(\xi_N^{\xi_0}(2a_{2N}) \cap A_2 \neq \emptyset) \geq \beta,$$

for all $\xi_0 \in \{0,1\}^{[1,N]}$, $\xi_0 \neq \emptyset$. By the Strong Markov property, we have the desired bound.

□

Now we return to the first term in (2.3.9). Since $S_{2kN-1}^{\zeta_0}$ is larger than $2(kN-2)a_{2N}$, given the information until this time, the event $\{S_{2kN-1}^{\zeta_0} \leq s_k - a_{2N}; C_k\}$ involves information between the times $(2kN-2)a_{2N}$ and $(2(k+1)N-2)a_{2N}$. Therefore, by the Strong Markov property and Claim 2.1 we conclude that

$$\mathbb{P}(\{\tau_N^{\zeta_0} > 2N^2 a_{2N}\} \cap \bigcap_{k=1}^{N} \{S_{2kN-1}^{\zeta_0} \leq s_k - a_{2N}\} \cap C_k) \leq \beta^N \tag{2.3.13}$$

for every $\zeta \in \mathcal{C}$.

We now analyze the second term in the right member of (2.3.9). From the fact that S is of order N^2 and a_{2N} is of order N^3, we get that for N large enough

$$s_k - 2a_{2N} = (2kN-2)a_{2N} \leq s_k + S \leq (2(k+1)N-2)a_{2N} = s_{k+1} - a_{2N}.$$

From these calculations, we have that the k-th event in the intersection inside the probability involves information in the interval of time $[s_k - a_{2N}, s_{k+1} - a_{2N}]$. Hence, the Markov property and Proposition 2.1 imply that this probability is less than $(1-\hat{\eta})^N$. Thus, putting together this last comment with (2.3.7), (2.3.8), (2.3.9), (2.3.13) and selecting N large enough such that

$$4\left(\frac{N}{2}c^N+\beta^N+(1-\hat{\eta})^N\right)\leq(2\max\{c,\beta,1-\hat{\eta}\})^N,$$

we obtain (2.3.6) for $\nu=2\max\{c,\beta,1-\hat{\eta}\}$. Item (ii) is immediate from the selection of c_N and d_N. □

In the rest of this chapter $c_N=2N^2a_{2N}$. From (2.3.5) and (2.3.6) we obtain a stronger result than item (i) of Proposition 2.4: there exists ν, $0<\nu<1$ such that for N large enough

$$\sup_{\zeta_0\in\mathcal{C}}\mathbb{P}(\zeta_{c_N}^{\mathbf{1},\mathbf{2},N}\neq\zeta_{c_N}^{\zeta_0,N};\tau_N^{\zeta_0}\geq c_N)\leq\nu^N. \tag{2.3.14}$$

Proof of Theorem 2.1. Let β_N as in the statement of Theorem 1. We will prove that

$$\lim_{N\to\infty}\left|\mathbb{P}(\tau_N^{\mathbf{1},\mathbf{2}}>\beta_N(t+s))-\mathbb{P}(\tau_N^{\mathbf{1},\mathbf{2}}>\beta_Nt)\mathbb{P}(\tau_N^{\mathbf{1},\mathbf{2}}>\beta_Ns)\right|=0, \tag{2.3.15}$$

which by the definition of β_N will imply

$$\lim_{N\to\infty}\mathbb{P}(\tau_N^{\mathbf{1},\mathbf{2}}\geq\beta_Nt)=e^{-t}.$$

To obtain the limit (2.3.15), we prove that there exist two positive sequences h_N and h'_N, both converging to zero when N goes to infinity, such that

$$\mathbb{P}(\tau_N^{\mathbf{1},\mathbf{2}}>\beta_Nt)\mathbb{P}(\tau_N^{\mathbf{1},\mathbf{2}}>\beta_Ns)-h_N\leq\mathbb{P}(\tau_N^{\mathbf{1},\mathbf{2}}>\beta_N(t+s)) \tag{2.3.16}$$

and

$$\mathbb{P}(\tau_N^{\mathbf{1},\mathbf{2}}>\beta_N(t+s))\leq\mathbb{P}(\tau_N^{\mathbf{1},\mathbf{2}}>\beta_Nt)\mathbb{P}(\tau_N^{\mathbf{1},\mathbf{2}}>\beta_Ns)+h'_N. \tag{2.3.17}$$

We begin by proving equation (2.3.16). First, we observe that for all t,s positives we have that

$$\begin{aligned}&\{\tau_N^{\mathbf{1},\mathbf{2}}>\beta_Ns;\tau_N^{\mathbf{1},\mathbf{2},\beta_Ns}>\beta_Nt;\zeta_{\beta_Ns}^{\mathbf{1},\mathbf{2},N}=\zeta_{\beta_Ns-c_N,c_N}^{\mathbf{1},\mathbf{2},N};\zeta_{\beta_Ns-c_N,2c_N}^{\mathbf{1},\mathbf{2},N}=\zeta_{\beta_Ns,c_N}^{\mathbf{1},\mathbf{2},N}\}\\&\subset\{\tau_N^{\mathbf{1},\mathbf{2}}>\beta_N(t+s)\},\end{aligned} \tag{2.3.18}$$

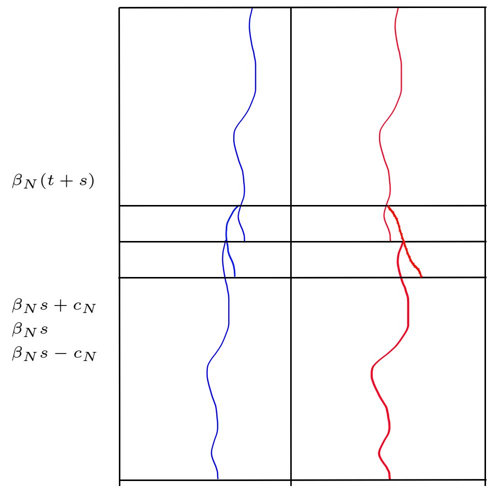

Figure 2.2: A graphic representation of the inclusion (2.3.18). Blue paths represent paths of particles 1 and red paths represent particles of type 2.

where $\zeta_{t,\cdot}^{\mathbf{1},\mathbf{2},N}$ and $\tau_N^{\mathbf{1},\mathbf{2},t}$ refer to the two-type contact process defined in $\Theta_{(0,t)}(\mathcal{H})$. By formula (2.3.18), we obtain that

$$\begin{aligned}&\mathbb{P}(\tau_N^{\mathbf{1},\mathbf{2}} > \beta_N s; \tau_N^{\mathbf{1},\mathbf{2},\beta_N s} > \beta_N t; \zeta_{\beta_N s}^{\mathbf{1},\mathbf{2},N} = \zeta_{\beta_N s - c_N, c_N}^{\mathbf{1},\mathbf{2},N}; \zeta_{\beta_N s - c_N, 2c_N}^{\mathbf{1},\mathbf{2},N} = \zeta_{\beta_N s, c_N}^{\mathbf{1},\mathbf{2},N})\\ &\leq \mathbb{P}(\tau_N^{\mathbf{1},\mathbf{2}} > \beta_N(t+s)).\end{aligned} \tag{2.3.19}$$

Now, we choose h_N as

$$\begin{aligned}h_N &= \mathbb{P}(\tau_N^{\mathbf{1},\mathbf{2}} > \beta_N s)\mathbb{P}(\tau_N^{\mathbf{1},\mathbf{2}} > \beta_N t)\\ &\quad - \mathbb{P}(\tau_N^{\mathbf{1},\mathbf{2}} > \beta_N s; \tau_N^{\mathbf{1},\mathbf{2},\beta_N s} > \beta_N t; \zeta_{\beta_N s}^{\mathbf{1},\mathbf{2},N} = \zeta_{\beta_N s - c_N, c_N}^{\mathbf{1},\mathbf{2},N}; \zeta_{\beta_N s - c_N, 2c_N}^{\mathbf{1},\mathbf{2},N} = \zeta_{\beta_N s, c_N}^{\mathbf{1},\mathbf{2},N}).\end{aligned}$$

Also, we observe that the Markov property implies that

$$\mathbb{P}(\tau_N^{\mathbf{1},\mathbf{2}} > \beta_N s)\mathbb{P}(\tau_N^{\mathbf{1},\mathbf{2}} > \beta_N t) = \mathbb{P}(\tau_N^{\mathbf{1},\mathbf{2}} > \beta_N s; \tau_N^{\mathbf{1},\mathbf{2};\beta_N s} > \beta_N t),$$

which gives

$$\begin{aligned} h_N &= \mathbb{P}(\tau_N^{\mathbf{1},\mathbf{2}} > \beta_N s; \tau_N^{\mathbf{1},\mathbf{2};\beta_N s} > \beta_N t; \zeta_{\beta_N s}^{\mathbf{1},\mathbf{2},N} \neq \zeta_{\beta_N s - c_N, c_N}^{\mathbf{1},\mathbf{2},N} \text{ or } \zeta_{\beta_N s - c_N, 2c_N}^{\mathbf{1},\mathbf{2},N} \neq \zeta_{\beta_N s, c_N}^{\mathbf{1},\mathbf{2},N}) \\ &\leq \mathbb{P}(\tau_N^{\mathbf{1},\mathbf{2}} > \beta_N s; \zeta_{\beta_N s}^{\mathbf{1},\mathbf{2},N} \neq \zeta_{\beta_N s - c_N, c_N}^{\mathbf{1},\mathbf{2},N}) + \mathbb{P}(\zeta_{\beta_N s - c_N, 2c_N}^{\mathbf{1},\mathbf{2},N} \neq \zeta_{\beta_N s, c_N}^{\mathbf{1},\mathbf{2},N}). \end{aligned} \tag{2.3.20}$$

Thus, by (2.3.20) and formula (2.3.14) we have that

$$h_N \leq 2\nu^N + \mathbb{P}(\tau_N^{\mathbf{1},\mathbf{2}} \leq 2c_N).$$

Therefore, h_N converges to zero when N goes to infinity. From this we deduce (2.3.16).

Now, to prove (2.3.17) we observe that by the Markov property and (2.3.14) we have that

$$\begin{aligned} &\mathbb{P}(\tau_N^{\mathbf{1},\mathbf{2}} > \beta_N(t+s)) \\ &\leq \mathbb{P}(\tau_N^{\mathbf{1},\mathbf{2}} > \beta_N t)\mathbb{P}(\tau_N^{\mathbf{1},\mathbf{2}} > \beta_N s) + \sup_{\zeta_0 \in \mathcal{C}} \mathbb{P}(\zeta_{\beta_N t}^{\mathbf{1},\mathbf{2},N} \neq \zeta_{\beta_N t}^{\zeta_0,N}; \tau_N^{\zeta_0} > \beta_N t)\mathbb{P}(\tau_N^{\mathbf{1},\mathbf{2}} > \beta_N s) \\ &\leq \mathbb{P}(\tau_N^{\mathbf{1},\mathbf{2}} > \beta_N t)\mathbb{P}(\tau_N^{\mathbf{1},\mathbf{2}} > \beta_N s) + \nu^N. \end{aligned}$$

Thus, we can take $h'_N = \nu^N$, and the proof is complete. □

2.4 Convergence in probability of $\frac{1}{N}\log(\tau_N^{\mathbf{1},\mathbf{2}})$

In this section, we prove Theorem 2.2, which states the asymptotic behavior of $\{\log \tau_N^{\mathbf{1},\mathbf{2}}\}_N$. Before the proof of the theorem, we present two technical results. Proposition below is a modification of Proposition 2.4 which is suitable for our purpose.

Proposition 2.5. *There exists* $0 < c < 1$ *such that for every* K

$$\sup_{\zeta_0 \in \mathcal{C}} \mathbb{P}(\tau_N^{\zeta_0} > 2N^2 K a_{2N}; \nexists\, t \leq 2N^2 K a_{2N} : \{x : \zeta_t^{\zeta_0,N}(x) = 2\} \subset [1,N]) \leq c^{KN} \tag{2.4.1}$$

for N *large enough.*

Proof. Let $s_k = k2NKa_{2N}$ for $1 \leq k \leq N$. We observe that the same argument used for the inclusion (2.3.5) leads to

$$\left\{\begin{array}{l} \tau_N^{\zeta_0} > 2N^2 K a_{2N}; \exists\, y_k \in A_2, z_k \in A_4 : \zeta_{s_k}^{\zeta_0,N}(y_k) = 2 \\ \zeta_{s_k}^{\zeta_0,N}(z_k) = 1 \text{ and } (y_k, s_k), (z_k, s_k) \text{ are an } N\text{-}barrier \end{array}\right\} \subset \{\{x : \zeta_{s_k+S}^{\zeta_0,N}(x) = 2\} \subset [1, N]\}. \tag{2.4.2}$$

Indeed, fix a configuration in the event on the left member of (2.4.2). Since (z_k, s_k) is an N-*barrier*, we have that if $\zeta_{s_k+S}^{\zeta_0,N}(x) \neq 0$ for a site $x \in [-N+1, 0]$, then $(x, s_k + S)$ is connected with (z_k, s_k) inside $[-N+1, 0]$ and by the priority of the particles of type 1 in $[-N+1, 0] \times [0, \infty)$, we have that $\zeta_{s_k+S}^{\zeta_0,N}(x) = 1$. By the same reasoning, we have that if $x' \in [1, N]$ and $\zeta_{s_k+S}^{\zeta_0,N}(x') \neq 0$, then $(x', s_k + S)$ is connected with (y_k, s_k) inside $[1, N]$ and by the priority of the particles of type 2 in $[1, N] \times [0, \infty)$, it holds that $\zeta_{s_k+S}^{\zeta_0,N}(x) = 2$. Summing up, at the time $s_k + S$ every site occupied in $[-N+1, 0]$ is occupied by a particle of type 1 and every site occupied in $[1, N]$ is occupied by a particle of type 2, which yields (2.4.2).

Now, we observe that (2.4.2) implies

$$\bigcup_{1 \le k \le N} \left\{\begin{array}{l} \tau_N^{\zeta_0} > 2N^2 K a_{2N}; \exists\, y_k \in A_2, z_k \in A_4 : \zeta_{s_k}^{\zeta_0,N}(y_k) = 2 \\ \zeta_{s_k}^{\zeta_0,N}(z_k) = 1 \text{ and } (y_k, s_k), (z_k, s_k) \text{ are an } N\text{-}barrier \end{array}\right\} \subset \bigcup_{0 \le t \le 2N^2 K a_{2N}} \{\{x : \zeta_{s_k+S}^{\zeta_0,N}(x) = 2\} \subset [1, N]\}. \tag{2.4.3}$$

Therefore, to conclude (2.4.1) it is enough to prove

$$\sup_{\zeta_0 \in \mathcal{C}} \mathbb{P}\left(\tau_N^{\zeta_0} > 2N^2 K a_{2N} \cap \bigcap_{k=1}^{KN} \left\{\begin{array}{l} \forall y \in A_2\ \zeta_{s_k}^{\zeta_0,N}(y) \neq 2 \text{ or } (y, s_k) \\ \text{is not an } N\text{-}barrier \end{array}\right\}\right) \le c^{KN}. \tag{2.4.4}$$

We observe that the left member in the equation above is the same as the left member of (2.3.8), with the only difference that in this case we are intersecting KN events instead of N. Thus, the same procedure used to get the bound c^N for the left member of (2.3.8) can be applied to obtain (2.4.4) (see Proposition 2.4). □

In the next lemma, we use the following limit

$$\lim_{N \to \infty} \frac{1}{N} \log(\mathbb{P}(T^{[1,N]} < \infty)) = -c_\infty, \tag{2.4.5}$$

where c_∞ is as in Remark 2.2. This result is proved for $R = 1$ in Lemma 3 of [9]. Since every step of this proof can be applied for the case $R > 1$, we assume (2.4.5) without proving it.

Lemma 2.1. *There exists $\theta > 0$ such that*

$$\liminf_{N\to\infty} \frac{1}{N} \log(\mathbb{P}(T^{[1,N]} < \theta N)) \geq -c_\infty.$$

Proof. Observe that for any $\theta > 0$

$$\mathbb{P}(T^{[1,N]} < \infty) = \mathbb{P}(T^{[1,N]} < \theta N) + \mathbb{P}(\theta N < T^{[1,N]} < \infty).$$

Using (2.2.6) for $t = \theta N$ we have

$$\mathbb{P}(T^{[1,N]} < \infty) \leq e^{-\theta N \hat{c}} + \mathbb{P}(T^{[1,N]} < \theta N). \tag{2.4.6}$$

By (2.4.5) and (2.4.6), for all $\epsilon > 0$ there exists an n such that for all $N > n$

$$e^{-(c_\infty+\epsilon)N} - e^{-\theta N \hat{c}} \leq \mathbb{P}(T^{[1,N]} < \theta N),$$

which implies

$$-(c_\infty + \epsilon)N + \log(1 + e^{-(\theta\hat{c} - c_\infty + \epsilon)N}) \leq \log \mathbb{P}(T^{[1,N]} < \theta N).$$

Taking $\theta > c_\infty/\hat{c}$, for every $\epsilon > 0$ we have

$$-(c_\infty + \epsilon) \leq \liminf_{N\to\infty} \frac{1}{N} \log \mathbb{P}(T^{[1,N]} < \theta N).$$

□

Proof of Theorem 2.2. First, for a fixed $\epsilon > 0$ we will prove that

$$\lim_{N\to\infty} \mathbb{P}(\tau_N^{\mathbf{1,2}} > k_N e^{(c_\infty+\epsilon)N}) = 0, \tag{2.4.7}$$

where $k_N = 2NK^* a_{2N} + \theta N$, $K^* = \left\lfloor \frac{2c_\infty + 2\epsilon}{\log(1/c)} \right\rfloor$, a_{2N} is as in Proposition 2.2, θ is as in Lemma 2.1 and c is as in Proposition 2.5.

To do this, we observe that by the Markov property, for every $n \in \mathbb{N}$ it holds that

$$\mathbb{P}(\tau_N^{\mathbf{1},\mathbf{2}} > nk_N) \leq (\sup_{\zeta \in \mathcal{C}} \mathbb{P}(\tau_N^{\zeta} > k_N))^n. \tag{2.4.8}$$

Now, we observe that by (2.4.1) for every $\zeta_0 \in \mathcal{C}$ we have

$$\begin{aligned} \mathbb{P}(\tau_N^{\zeta_0} > k_N) \leq c^{K^*N} + \\ + \mathbb{P}(\tau_N^{\zeta_0} > k_N; \exists\, t \leq k_N - \theta N : \{x : \zeta_t^{\zeta_0,N}(x) = 2\} \subset [1,N]). \end{aligned} \tag{2.4.9}$$

Furthermore, the strong Markov property and the atractiveness of the classic contact process give that

$$\begin{aligned} \mathbb{P}(\tau_N^{\zeta_0} > k_N; \exists\, t \leq k_N - \theta N : \{x : \zeta_t^{\zeta_0,N}(x) = 2\} \subset [1,N]) \\ \leq \mathbb{P}(T^{[1,N]} > \theta N), \end{aligned} \tag{2.4.10}$$

and by Lemma 2.1, for N large enough we have

$$\mathbb{P}(T^{[1,N]} > \theta N) \leq 1 - e^{-(c_\infty + 2\epsilon)N}. \tag{2.4.11}$$

Substituting formulas (2.4.10) and (2.4.11) into (2.4.9) we obtain that

$$\sup_{\zeta_0 \in \mathcal{C}} \mathbb{P}(\tau_N^{\zeta_0} > k_N) \leq c^{K^*N} + 1 - e^{-(c_\infty + 2\epsilon)N}. \tag{2.4.12}$$

Thus, by (2.4.8) and (2.4.12), for $N^* = \lfloor e^{(c_\infty + \epsilon)N} \rfloor$ we have

$$\mathbb{P}(\tau_N^{\zeta_0} > N^* k_N) \leq (1 - e^{-(c_\infty + 2\epsilon)N} + e^{-K^* \log(1/c)N})^{N^*}.$$

Now, by our choice of K^*, we conclude (2.4.7). Moreover, observe that (2.4.7) implies

$$\lim_{N \to \infty} \mathbb{P}(\frac{1}{N} \log(\tau_N^{\zeta_0}) > c_\infty + \epsilon) = 0. \tag{2.4.13}$$

To conclude the proof, we only need to state that for every $\epsilon > 0$

$$\lim_{N \to \infty} \mathbb{P}(\frac{1}{N} \log(\tau_N^{\zeta_0}) < c_\infty - \epsilon) = 0. \tag{2.4.14}$$

For this purpose, observe that $\tau_N^{\mathbf{1},\mathbf{2}}$ is stochastically larger than the minimum of two independent variables with the same law of $T_N^{\mathbf{1}}$. Therefore, we have that

$$\begin{aligned}\mathbb{P}(\frac{1}{N}\log(\tau_N^{\zeta_0}) < c_\infty - \epsilon) &\leq \mathbb{P}(\frac{1}{N}\log(\min\{T_N^{\mathbf{1}};\tilde{T}_N^{\mathbf{1}}\}) < c_\infty - \epsilon)\\ &= \mathbb{P}(\min\{T_N^{\mathbf{1}};\tilde{T}_N^{\mathbf{1}}\} < e^{(c_\infty-\epsilon)N})\\ &= \mathbb{P}(T_N^{\mathbf{1}} < e^{(c_\infty-\epsilon)N})^2,\end{aligned} \tag{2.4.15}$$

where $T_N^{\mathbf{1}}$ and $\tilde{T}_N^{\mathbf{1}}$ are i.i.d. By (2.2.5), the limit of the last term in (2.4.15) is zero, which implies (2.4.14).

Clearly, from (2.4.13) and (2.4.14) the theorem follows. □

In the next remark, we discuss what happens after the first type is extinguished. During this remark, we denote by $\xi^A_{[-N+1,N]}(t)$ the classic contact process with initial configuration A and $T^A_{[-N+1,N]}$ the time of extinction of this process. For the special case $A = [-N+1, N]$, we write $\xi^{\mathbf{1}}_{[-N+1,N]}(t)$ and $T^{\mathbf{1}}_{[-N+1,N]}$.

Remark 2.3. *Let $\tilde{T}^{\mathbf{1}}_{2N}$ be the time of the extinction of both particles, that is*

$$\tilde{T}^{\mathbf{1}}_{2N} = \inf\{t > 0 : \zeta_t^{\mathbf{1},\mathbf{2},N} = \emptyset\}.$$

If we ignore the existence of both types of particles, the dynamic of the process is the same as the classic contact process. Therefore, $\tilde{T}^{\mathbf{1}}_{2N}$ has the same distribution as $T^{\mathbf{1}}_{[-N+1,N]}$ and, consequently, Remark 2.2 implies that for $\epsilon > 0$ we have

$$\lim_{N\to\infty} \mathbb{P}\left(e^{(c_\infty-\epsilon)2N} < \tilde{T}^{\mathbf{1}}_{2N} < e^{(c_\infty+\epsilon)2N}\right) = 1. \tag{2.4.16}$$

Moreover, observe that after $\tau_N^{\mathbf{1},\mathbf{2}}$ the process behaves like the classic contact process, since after that time there is only one type of particle. Observe also that combining (2.4.16) *with Theorem 2.2 we obtain*

$$\lim_{N\to\infty} \mathbb{P}\left(e^{(c_\infty-\epsilon)2N} - e^{(c_\infty+\epsilon)N} < \tilde{T}^{\mathbf{1}}_{2N} - \tau_N^{\mathbf{1},\mathbf{2}}\right) = 1. \tag{2.4.17}$$

Furthermore, after the extinction of the first species, the species that survives behaves like the classic contact process. Thus, $\tilde{T}^{\mathbf{1}}_{2N} - \tau_N^{\mathbf{1},\mathbf{2}}$ has the same law of $T^{A_N}_{[-N+1,N]}$, where

$A_N = \zeta^{\mathbf{1},\mathbf{2},N}_{\tau^{\mathbf{1},\mathbf{2}}_N}$. *Using* (**??**) *we have that*

$$\lim_{N\to\infty} \mathbb{P}(\xi^{A_N}_{a_{2N}} = \xi^{\mathbf{1}}_{[-N+1,N]}(a_{2N}); T^{A_N}_{[-N+1,N]} > a_{2N}) = 0$$

and by (2.4.17) *and the limit above we have that*

$$\lim_{N\to\infty} \mathbb{P}(\xi^{A_N}_{a_{2N}} = \xi^{\mathbf{1}}_{[-N+1,N]}(a_{2N})) = 0.$$

Therefore

$$\lim_{N\to\infty} \mathbb{P}(T^{A_N}_{[-N+1,N]} = T^{\mathbf{1}}_{[-N+1,N]}) = 1$$

and by Remark 2.2 we obtain that

$$\lim_{N\to\infty} \frac{1}{2N} \log(\tilde{T}^{\mathbf{1}}_{2N} - \tau^{\mathbf{1},\mathbf{2}}_N) = \lim_{N\to\infty} \frac{1}{2N} \log T^{\mathbf{1}}_{[-N+1,N]} = c_\infty \text{ in Probability.}$$

Chapter 3

The contact process on the layer $[-L, L] \times \mathbb{Z}$

In this chapter we prove that the contact process restricted to $[-L, L]\times\mathbb{Z}$, with two types of particles, the particles of type 1 having priority in $[-L, L] \times (-\infty, 0]$ and the particles of type 2 in $[-L, L] \times [1, \infty)$, presents metastability behavior. We restrict our study to the case where the range is $R = 1$. To this aim, we will need a main tool, an extension of Mountford-Sweet renormalization to the contact process in $[-L, L]\times\mathbb{Z}$, presented in Section 3.1. This construction is the key tool to extend, for the contact process in $[-L, L] \times \mathbb{Z}$, the notion of "barriers" introduced in Section 2.1. Once we have the definition of "barriers", the proofs follow closely to those in Sections 2.2 and 2.3 to obtain the equivalents results for the contact process in $[-L, L] \times \mathbb{Z}$.

3.1 An extension of Mountford-Sweet renormalization

In Section 1.4 we described a *k-dependent* percolation system, Ψ, with closure close to 0, as introduced in [16]. In this section we define an equivalent percolation system for the contact process restricted to the layer $[-L, L] \times \mathbb{Z}$, with infection parameter $\lambda > \lambda_L$, where λ_L is the critical parameter in $[-L, L] \times \mathbb{Z}$, defined in (1.5.1).

To simplify notation, during this section we refer to the contact process restricted to $B_L = [-L, L] \times \mathbb{Z}$, with initial configuration $A \subset [-L, L] \times \mathbb{Z}$, as $\xi^A_{B_L}(t)$. In the special case $A = B_L$ we write $\xi^{\mathbb{1}}_{B_L}(t)$.

Given $(m,n) \in \Lambda$ we define

$$\mathcal{I}_m^{\tilde{N}} = \left([-L,L] \times \left(\frac{m\tilde{N}}{2} - \frac{\tilde{N}}{2}, \frac{m\tilde{N}}{2} + \frac{\tilde{N}}{2} \right] \right) \cap \mathbb{Z}^2,$$
$$I_{(m,n)}^{\tilde{N}} = \mathcal{I}_m^{\tilde{N}} \times \{3\tilde{N}n\},$$
$$J_{(m,n)}^{\tilde{N}} = [-L,L] \times \left\{ \frac{m\tilde{N}}{2} \right\} \times [3\tilde{N}n, 3\tilde{N}(n+1)].$$

We define a map $\Psi_L(\mathcal{H}) : \Lambda \to \{0,1\}$ as follows: set $\Psi_L(m,n) = 1$ if all the conditions below are satisfied

$$\begin{gathered}\text{For each rectangle } [-L,L] \times I \subset \mathcal{I}_{m-1}^{\tilde{N}} \cup \mathcal{I}_{m+1}^{\tilde{N}} \text{ of area } (2L+1)\sqrt{\tilde{N}}, \\ \text{we have } [-L,L] \times I \cap \xi_{B_L}^{\mathbb{1}}(3\tilde{N}(n+1)) \neq \emptyset;\end{gathered} \tag{3.1.1}$$

$$\begin{gathered}\text{If } z \in \mathcal{I}_{m-1}^{\tilde{N}} \cup \mathcal{I}_{m+1}^{\tilde{N}} \text{ and } \xi_{B_L}^{\mathbb{1}}(3\tilde{N}n) \times \{3\tilde{N}n\} \to (z, 3\tilde{N}(n+1)), \text{ then} \\ (\xi_{B_L}^{\mathbb{1}}(3\tilde{N}n) \times \{3\tilde{N}n\}) \cap I_{(m,n)}^{\tilde{N}} \to (z, 3\tilde{N}(n+1));\end{gathered} \tag{3.1.2}$$

$$\begin{gathered}\text{If } (z,s) \in J_{(m,n)}^{\tilde{N}} \text{ and } \xi_{B_L}^{\mathbb{1}}(3\tilde{N}n) \times \{3\tilde{N}n\} \to (z,s), \\ \text{then } (\xi_{B_L}^{\mathbb{1}}(3\tilde{N}n) \times \{3\tilde{N}n\}) \cap I_{(m,n)}^{\tilde{N}} \to (z,s);\end{gathered} \tag{3.1.3}$$

$$\begin{aligned}&\left\{ \begin{array}{c} z' \in [-L,L] \times \mathbb{Z} : \exists s,t, 3\tilde{N}n \le s < t \le 3\tilde{N}(n+1), \\ z \in \mathcal{I}_{m-1}^{\tilde{N}} \cup \mathcal{I}_{m+1}^{\tilde{N}} \text{ such that } (z',s) \to (z,t) \end{array} \right\} \\ &\qquad\qquad \subset [-L,L] \times \left[\frac{m\tilde{N}}{2} - c_1 3\tilde{N}, \frac{m\tilde{N}}{2} + c_1 3\tilde{N} \right],\end{aligned} \tag{3.1.4}$$

where c_1 is a constant that will be specified below. Set $\Psi_L(m,n) = 0$ otherwise.

We dedicate this section to explain how we choose the parameter $\tilde{N}$ such that the law of Ψ_L is a finite dependent percolation system with closure close to 0.

The first step is to determine the constant c_1 in (3.1.4). We define the rightmost second coordinate as follows

$$\mathbf{r}_s = \max\{y : \exists\, x \in [-L,L] \text{ such that } (x,y) \in \xi_{B_L}^{[-L,L]\times(-\infty,0]}(s)\},$$

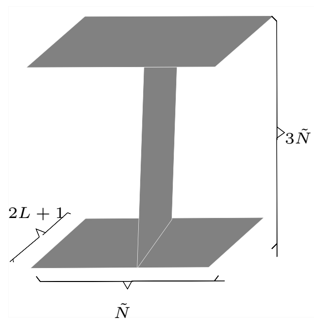

Figure 3.1: Renormalized site for Mountford-Sweet extension

and the leftmost second coordinate

$$\mathbf{l}_s = \min\{y : \exists\, x \in [-L, L] \text{ such that } (x, y) \in \xi_{B_L}^{[-L,L]\times[1,\infty)}(s)\}.$$

Lemma 3.1. *There exist c_1 and c_2 positive constants such that*

$$\mathbb{P}\left(\inf_{0\le s\le t} \mathbf{l}_s < -c_1 t\right) = \mathbb{P}\left(\sup_{0\le s\le t} \mathbf{r}_s > c_1 t\right) \le e^{-c_2 t}. \tag{3.1.5}$$

Proof. It is clear that $\mathbf{r}_t$ is bounded by a Poisson process N_t with rate $(2L+1)\lambda$, therefore

$$\begin{aligned}\mathbb{P}(N_t > 2(2L+1)\lambda t(e-1)) &= \mathbb{P}(e^{N_t} > e^{2(2L+1)t\lambda(e-1)}) \\ &\le \mathbb{E}(e^{N_t})e^{-2(2L+1)t\lambda(e-1)} = e^{(2L+1)\lambda t(e-1)}e^{-2(2L+1)\lambda t(e-1)} \\ &= e^{-(2L+1)\lambda t(e-1)},\end{aligned}$$

where the inequality follows by Markov inequality and the second equality is the formula of the generating function for the Poisson distribution with mean $(2L+1)\lambda t$. Therefore

$$\mathbb{P}\left(\sup_{0\le s\le t} \mathbf{r}_s > 2(2L+1)\lambda(e-1)t\right) \le \mathbb{P}(N_t > 2(2L+1)\lambda(e-1)t) \le e^{-(2L+1)\lambda(e-1)t},$$

and we obtain the lemma for $c_1 = 2(2L+1)\lambda(e-1)$ and $c_2 = (2L+1)\lambda(e-1)$. □

The next three results are proved in [16] for the contact process in dimension 1 with finite range larger than 1. The arguments on those proofs are immediately extended for the contact process in the layer $[-L,L]\times\mathbb{Z}$ only that substituting the Bezuidenhout-Grimmett renormalization for the adaptation that we mentioned in Section 1.5. For the seek of completeness, we rewrite these results for our case. For details see [16].

Proposition 3.1. *Let $\xi_{B_L}(t)$ be the contact process in $[-L,L]\times\mathbb{Z}$ with rate of infection $\lambda>\lambda_L$. For any $x\in[-L,L]$ the probability that the site $(x,0)$ survives to time t inside the layer $[-L,L]\times\mathbb{Z}$ and that there exists a vacant rectangle $[-L,L]\times I$ at time t of length $(2L+1)d\sqrt{t}$ inside $[-L,L]\times[-t/2,t/2]$ is bounded by $Ce^{-cd\sqrt{t}}$, where c and C are positive finite constants independent of $d\in(0,1]$ and t.*

Idea of the proof: Let T as in Proposition 1.2. We denote the event in the statement of the proposition as

$$C=\left\{\begin{array}{c}\xi_{B_L}^{(x,0)}(t)\neq\emptyset;\exists \text{ an empty rectangle } [-L,L]\times I\\ \text{of area } d(2L+1)\sqrt{t} \text{ inside } [-L,L]\times[-\frac{1}{2}t,\frac{1}{2}t]\end{array}\right\}, \tag{3.1.6}$$

and also we define

$$D=\left\{\begin{array}{c}\Psi_{B_L}^{(x,0)}(t)\neq\emptyset;\exists \text{ a rectangle } [-L,L]\times I \text{ of area } d(2L+1)\sqrt{t}\\ \text{inside } [-L,L]\times[-\frac{1}{2}t,\frac{1}{2}t] \text{ which remains empty throughout the time interval } [t,t+24T]\end{array}\right\}. \tag{3.1.7}$$

We can have that

$$\mathbb{P}(C)e^{-2(L+1)R^2 24T}\leq\mathbb{P}(D),$$

therefore, instead of proving the estimate for the probability of C we are going to do it for the probability of D. The idea will be to prove that a vacant rectangle in the contact process of this area implies a vacant interval in a certain percolation system constructed using the Bezuidenhout-Grimmett renormalization. As a consequence of Corollary 1.1 it is possible to prove that, outside an event with probability less than $e^{-cd\sqrt{k}}$ for k large enough, if $\Phi^{\{0\}}$ survives until time k then does not exist an empty interval of length $d\sqrt{k}$ inside $[-\hat{\beta}k,\hat{\beta}k]$, where $\hat{\beta}$ is the percolation edge speed for Φ.

We take $\delta=\frac{\hat{\beta}}{48Tc_1+2\hat{\beta}}$ such that

$$[-t/2,t/2]\subset\bigcup_{m=-\hat{\beta}n}^{\hat{\beta}n}(\mathcal{R}^{\pm}_{(m,n)}+(z,s)), \tag{3.1.8}$$

for $s \in [0, \delta t]$, $z \in [-L, L] \times [-c_1\delta t, c_1\delta t]$ and $n = \lceil (t-\delta t)/24T \rceil$. It is possible to prove that outside an event with probability smaller than $e^{-c\sqrt{\delta t}}$ there exists a stopping time $\mathcal{T} \in [0, \delta t]$ and a site $z(\mathcal{T})$ such that $\Phi_{(z(\mathcal{T}),\mathcal{T})}$ survives for all times. By Lemma 3.1 we can also suppose that in this event $z(\mathcal{T}) \in [-c_1\delta t, c_1\delta t]$. Select N such that $\mathcal{T} + 24T \in [t, t + 24T]$, then by the arguments above

$$\begin{aligned}
&\mathbb{P}(D) \le e^{-cd\sqrt{\delta t}} \\
&+ \sum_{N=\lceil (1-\delta)t/24T \rceil}^{\lceil t/24T \rceil} \mathbb{P}(\Phi^{\{0\}}_{z(\mathcal{T}),\mathcal{T}}(N) \neq \emptyset, \exists \text{ a vacant interval of length } \frac{d\sqrt{t}}{34K} \text{ in } [-\hat{\beta}N, \hat{\beta}N]) \\
&\le e^{-cd\sqrt{\delta t}} \\
&+ \sum_{N=\lceil (1-\delta)t/24T \rceil}^{\lceil t/24T \rceil} \mathbb{P}(\Phi^{\{0\}}_{z(\mathcal{T}),\mathcal{T}}(N) \neq \emptyset, \exists \text{ a vacant interval of length } dc'\sqrt{N} \text{ in } [-\hat{\beta}N, \hat{\beta}N]) \\
&\le e^{-cd\sqrt{\delta t}} + e^{-c''d\sqrt{t}}
\end{aligned}$$

with $0 < c' \le 1$ and c'' independent on d and t.

Remark 3.1. *Using Corollary 1.1 for the Benzuidenhout-Grimmett renormalization, it is possible to prove the following: for any k and m there is a constant $c > 0$, depending on k and m, such that if on the rectangle $[-L, L] \times [0, N]$ both A and B are subsets that intersect every interval of area $m\sqrt{N}$, then $\mathbb{P}(\xi_t^B = 0 \text{ on } A) \le e^{-c\sqrt{N}}$ for every $t \in [kN, 2kN]$ and sufficiently large N.*

In [16] is stated the equivalent result for the contact process in dimension 1 and finite range.

Lemma 3.2. *Let A be a subset of $[-L, L] \times \mathbb{Z}$ such that A intersects every rectangle $[-L, L] \times I$ of area $(2L+1)\sqrt{N}$ inside $[-L, L] \times [0, N]$, then for every $z \in [-L, L] \times [-N/2, 3N/2]$ we have that*

$$\mathbb{P}(\xi^{\mathbb{1}}_{B_L}(3N)(z) = 1, \xi^{A\cap([-L,L]\times[0,N])}_{B_L}(3N)(z) = 0) \le Ce^{-cN^{1/2}}. \tag{3.1.9}$$

Proof. For $z = (z_1, z_2) \in [-L, L] \times [-N/2, 3N/2]$, denote $\tilde{\xi}^{(z,3N)}_{B_L}(t)$ the dual contact process restricted to $[-L, L] \times \mathbb{Z}$ and beginning in $(z, 3N)$. By Proposition 3.1 we have that at time $2N$ the probability that this process has an empty rectangle $[-L, L] \times I$ of area

$\sqrt{3N}/\sqrt{3} = \sqrt{N}$ in $[-L, L] \times [-N, N] + z$ is smaller than $Ce^{c\sqrt{3N}}$. By our choice of z we have that

$$[-L, L] \times [-N, N] + z \supset [-L, L] \times [0, N].$$

Therefore, outside an event with probability smaller than $Ce^{c\sqrt{3N}}$, $\tilde{\xi}_{B_L}^{(z,3N)}(t)$ intersects every rectangle $[-L, L] \times I$ of area $\sqrt{N}$ inside $[-L, L] \times [0, N]$.

By Remark 3.1 there is a constant $c > 0$ such that if on the rectangle $[-L, L] \times [0, N]$ both A and B are subsets of $[-L, L] \times \mathbb{Z}$ that intersect every rectangle $[-L, L] \times I$ of area $\sqrt{N}$, then

$$\mathbb{P}(\xi^A_{[-L,L]\times[0,N]} = 0 \text{ on } B) \leq e^{-c\sqrt{N}},$$

for sufficiently large N. Thus, the probability that $\xi_{B_L}^{A\cap([-L,L]\times[0,N])}(N) = 0$ on $\tilde{\xi}_{B_L}^{(z,3N)}(2N) \cap ([-L, L] \times [0, N])$ is smaller than $Ce^{c\sqrt{3N}}$. □

Lemma 3.3. *Let A be a set of $[-L, L] \times [0, N]$ that intersects every rectangle $[-L, L] \times I$ of area $(2L + 1)\sqrt{N}$. The probability that*

$$\{\xi^A_{B_L}(s)(z) = 0; \xi_{B_L}^{[-L,L]\times([0,N]^c)}(s)(z) = 1\}, \tag{3.1.10}$$

for some $z \in [-L, L] \times \{N/2\}$ and $s \in [0, 3N]$, is less than $Ce^{-c\sqrt{N}}$,where C and c positive constants independent of N.

Proof. We rewrite the proof of Corollary 5 in [16] for our case:

It is clear that we can prove the statement in the lemma for a fixed z because there is a bounded quantity of sites. Thus, for a given $z \in [-L, L] \times \{N/2\}$ denote the event in (3.1.10) by $B(s)$. For a fixed configuration in the union of $B(s)$, $0 \leq s \leq 3N$ denote s_0 as the first time such that $B(s_0)$ occurs. Since s_0 is a stopping time we have that given s_0 with probability equal to $e^{-(4\lambda+1)}$ there are no marks of infection coming in or marks of death at z during the time interval $[s_0, s_0 + 1]$, for any $z \in [-L, L] \times \{N/2\}$. Therefore

$$\mathbb{P}(\bigcup_s B(s)) \leq e^{(4\lambda+1)} \sum_{1\leq k\leq 3N} \mathbb{P}(B(k)).$$

Hence, it is enough to prove for a fix (z, s) that the event in (3.1.10) has probability less than $Ce^{-c\sqrt{N}}$.

We divided into two cases. First case if $s \leq N/2c_1$, for c_1 as in Lemma 3.1. Using

Lemma 3.1 for the dual process beginning at (z,s) we obtain that the event in (3.1.10) has probability less than $Ce^{-c\sqrt{N}}$.

Now, suppose that $s \geq N/2c_1$. Suppose that for the point (z,s) with $z \in [-L,L]\times\{N/2\}$ we have (3.1.10). Then the dual process beginning at (z,s) survives for a time $s \geq N/c_1$. Denote $\{\tilde{\xi}^{(z,s)}_{B_L}(t)\}_{0\leq t\leq s}$ as the dual contact process restricted to $[-L,L]\times\mathbb{Z}$. By Proposition 3.1, outside an event with probability less than $Ce^{-cd\sqrt{t}}$, $\tilde{\xi}_{B_L}(t)$ intersects every rectangle $[-L,L]\times I$ of area $(2L+1)d\sqrt{t}$ in $[-L,L]\times([-t/2,t/2]+N/2)$, with $t \leq s$. This implies that, outside an event with probability less than $Ce^{-cd\sqrt{N/4c_1}}$, $\tilde{\xi}^{(z,s)}_{B_L}(s-N/4c_1)$ intersects every rectangle $[-L,L]\times I$ of area $d\sqrt{s-N/4c_1}$ in $[-L,L]\times[(1/2-1/8c_1)N,(1/2+1/8c_1)N]$.

If $s \leq N+N/4c_1$, then $s-N/4c_1 \leq N$ and $\tilde{\xi}^{(z,s)}_{B_L}(s-N/4c_1)$ intersects every rectangle $[-L,L]\times I$ of area $\sqrt{N}$ in $[-L,L]\times[(1/2-1/8c_1)N,(1/2+1/8c_1)N]$ (outside an event with probability smaller than $Ce^{-cd\sqrt{N/4c_1}}$).

If $s > N+N/4c_1$ we take $d=\sqrt{N}/\sqrt{s-N/4c_1}$ and again we obtain that $\tilde{\xi}^{(z,s)}_{B_L}(s-N/4c_1)$ intersects every rectangle $[-L,L]\times I$ of area $\sqrt{N}$ in $[-L,L]\times[(1/2-1/8c_1)N,(1/2+1/8c_1)N]$ (outside an event with probability smaller than $Ce^{-cd\sqrt{N/4c_1}}$).

In any case, by Remark 3.1 we have that the probability of the event in (3.1.10) is smaller than $Ce^{-c\sqrt{N}}$. □

For $\delta > 0$ we choose $\tilde{N}$ such that

The event in Proposition 3.1 has probability less than $\delta/4$ for $t = 3\tilde{N}$;

Probability (3.1.9) in Lemma 3.2 is less than $\delta/4$;

The event in Lemma 3.3 has probability less than $\delta/4$;

The probability in Lemma 3.1 is less than $\delta/8$ for $t = 3\tilde{N}$.

These restrictions on $\tilde{N}$ imply that given the Harris construction until time $3\tilde{N}n$ the probability of the event the simultaneous occurrence of (3.1.1), (3.1.2), (3.1) and (3.1.4) is larger than $(1-\delta)$. Therefore, for a suitable k depending on $\tilde{N}$, the map Ψ_L is a k-dependent oriented percolation system with closure smaller than δ.

3.2 Barriers and Metastability

Let $\zeta_L^{\mathbf{1},\mathbf{2},N}(t)$ be the contact process restricted to the layer $[-L,L]\times[-N+1,N]$ at time t, initial configuration $\mathbb{1}_{[-L,L]\times[-N+1,0]}+2\mathbb{1}_{[-L,L]\times[1,N]}$ and the particles of type 1 have priority in $[-L,L]\times[-N+1,0]$ and the particles of type 2 in $[-L,L]\times[1,N]$. In this section we prove that the time of the first extinction $\tau_N^{L,\mathbf{1},\mathbf{2}}$ properly rescaled converges to the exponential distribution with rate 1. The procedure to obtain this result will be to adapt every step used in the proof of the case $R>1$ and $d=1$.

First we extend the definition of *N-barrier* **??** to the contact process in the layer $[-L,L]\times\mathbb{Z}$. For $\tilde{N}$ as in Mountford-Sweet extension, let us denote $\tilde{M}=\tilde{M}(N)=\lfloor\frac{2(N-2c_1\tilde{K}\tilde{N})}{\tilde{N}}\rfloor$ and we set $\tilde{S}=\tilde{S}(N)=\tilde{K}\tilde{N}\tilde{M}^2+2$.

***Definition* 3.1.** *(a) For $z\in[-L,L]\times[1,N]$ we say $(z,0)$ is an $L\times N$-barrier if for all $z'\in[-L,L]\times[1,N]$ such that $[-L,L]\times\mathbb{Z}\times\{0\}\to(z',\tilde{S})$, then $(z,0)\to(z',\tilde{S})$ inside $[-L,L]\times[1,N]$.*

(b) For $z\in[-L,L]\times[-N+1,0]$ we say $(z,0)$ is an $L\times N$-barrier if for all $z'\in[-L,L]\times[-N+1,0]$ such that $[-L,L]\times\mathbb{Z}\times\{0\}\to(z',\tilde{S})$, then $(z,0)\to(z',\tilde{S})$ inside $[-L,L]\times[-N+1,0]$.

To prove the equivalent of Proposition 2.1 for our case, the argument is the same as in that scenario, only that substituting in the proof the Mountford-Sweet renormalization for the renormalization defined in Section 3.1. Therefore, we obtain that

Proposition 3.2. *There exists $\tilde{\eta}=\tilde{\eta}(\lambda,L)>0$ such that for all N large enough*

$$\mathbb{P}^z((z,0)\text{ is an }L\times N\text{-barrier})>\tilde{\eta}, \tag{3.2.1}$$

for any $z\in[-L,L]\times\left[\frac{-\tilde{M}\tilde{N}}{2}-\frac{\tilde{N}}{2},\frac{-\tilde{\imath}\tilde{N}}{2}+\frac{\tilde{N}}{2}\right]\cup[-L,L]\times\left[\frac{\tilde{\imath}\tilde{N}}{2}-\frac{\tilde{N}}{2},\frac{\tilde{M}\tilde{N}}{2}+\frac{\tilde{N}}{2}\right]$, where $\tilde{\imath}$ is given by

$$\tilde{\imath}=\tilde{\imath}(N)=\begin{cases}\lfloor 4\alpha\tilde{K}\tilde{N}\rfloor & \text{if } \tilde{M}^2+\lfloor 4\alpha\tilde{K}\tilde{N}\rfloor \text{ is even,}\\ \lfloor 4\alpha\tilde{K}\tilde{N}\rfloor+1 & \text{if } \tilde{M}^2+\lfloor 4\alpha\tilde{K}\tilde{N}\rfloor \text{ is odd.}\end{cases} \tag{3.2.2}$$

To simplify notation we denote

$$\tilde{A}_2=[-L,L]\times\left[\frac{-\tilde{M}\tilde{N}}{2}-\frac{\tilde{N}}{2},\frac{-\tilde{\imath}\tilde{N}}{2}+\frac{\tilde{N}}{2}\right],$$

$$\tilde{A}_4 = [-L,L] \times \left[\frac{\tilde{\imath}\tilde{N}}{2} - \frac{\tilde{N}}{2}, \frac{\tilde{M}\tilde{N}}{2} + \frac{\tilde{N}}{2}\right],$$

$$\tilde{A}_1 = [-L,L] \times \left[-N+1, \frac{-\tilde{M}\tilde{N}}{2} - \frac{\tilde{N}}{2} - 1\right], \ \tilde{A}_5 = [-L,L] \times \left[N, \frac{\tilde{M}\tilde{N}}{2} + \frac{\tilde{N}}{2}\right]$$

and

$$\tilde{A}_3 = \left[\frac{-\tilde{\imath}\tilde{N}}{2} + \frac{\tilde{N}}{2}, \frac{\tilde{\imath}\tilde{N}}{2} - \frac{\tilde{N}}{2}\right].$$

Now we need to obtain regeneration for the classic contact process in $[-L,L] \times [1,N]$. We proceed as in Proposition 2.2. By the result above given an occupied site, we have a positive probability of creating an $L \times N$-barrier. For a given configuration $\xi \in [-L,L] \times [1,N]$ such that the infection last until time $N(\tilde{S}+1)$ then outside an event with probability $(1-\tilde{\eta})^N$ there exists an $L \times N$-barrier in $\tilde{A}_2$. The definition of $L \times N$-barrier implies the coupling of the two processes, the one with initial configuration ξ and the one beginning with full occupancy in $[-L,L] \times [1,N]$. We denote $\tilde{a}_N = N(\tilde{S}+1)$ as the regeneration time for the classic contact process in the layer. We select $\tilde{b}_N = b_N$, for b_N as in item (iii) of Proposition 2.2.

The next step is the proof of the equivalent to Proposition 2.3 for the contact process in the layer. This result first gives an estimative for the stopping times S_k and $\hat{S}_k$. First of all we redefine this two stopping times to the contact process with two type of particles and priority in the layer $[-L,L] \times [-N,N]$. Given $k \geq 1$ and N, define the following stopping times

$$S_{L,k} = \inf\{t > (k-1)\tilde{a}_{2N} : \exists\, z \in [-L,L] \times [1,N],\ \zeta_L^N(t)(y) = 2\},$$

and

$$\hat{S}_{L,k} = \inf\{t > (k-1)\tilde{a}_{2N} : \exists\, z' \in [-L,L] \times [-N+1,0],\ \zeta_L^N(t)(z') = 1\}.$$

We remember to the reader that for Proposition 2.3 the main two ingredients were the regeneration for the contact process in dimension 1 and Lemma 1.1. For the contact process in the layer we have also the regeneration and Lemma 1.1, the rest of the arguments are also valid for this case. Therefore, we can state the proposition

Proposition 3.3. *There exists c, $0 < c < 1$, such that*

$$\sup_{\zeta \in \mathcal{C}_L} \mathbb{P}(\tau_N^\zeta > N\tilde{a}_{2N}; \exists\, k \;\; 1 \leq k \leq N : S_{L,k} > k\tilde{a}_{2N}) \leq N c^{2N}, \tag{3.2.3}$$

for all N large enough.

With the same arguments used in Proposition 2.4, Proposition 3.2 and Proposition 3.3 together implies the regeneration for the contact process with two type of particles and priorities in the layer. To explain the idea of the proof for the contact process in the layer first we define the following sites

$$z_k^\zeta \text{ such that } z_k^\zeta \in \tilde{A}_4 \text{ and } \zeta_L^{\zeta,N}(\tilde{c}_k)(z_k^\zeta) = 2;$$

and

$$\mathsf{z}_k^\zeta \text{ such that } \mathsf{z}_k^\zeta \in \tilde{A}_2 \text{ and } \zeta_L^{\zeta,N}(\tilde{c}_k)(\mathsf{z}_k^\zeta) = 1,$$

for a configuration $\zeta \in \mathcal{C}_L$, $1 \leq k \leq N$ and $\tilde{c}_k = 2kN\tilde{a}_{2N}$.

The idea of the proof is the same: by Proposition 3.3 we have that outside an event with exponential small probability if both type of particles are alive for the process with the initial configuration ζ and for the process with the initial configuration $\mathbb{1}_{[-L,L]\times[-N+1,0]} + 2\mathbb{1}_{[-L,L]\times[1,N]}$ until time $\tilde{c}_N$ we can define z_k^ζ, z_k^ζ, $z_k^{\mathbf{1,2}}$ and $\mathsf{z}_k^{\mathbf{1,2}}$. By Proposition 3.2 from each site $(\cdot, \tilde{c}_k)$, $\cdot \in \{z_k^\zeta, \mathsf{z}_k^\zeta, z_k^{\mathbf{1,2}}, \mathsf{z}_k^{\mathbf{1,2}}\}$, we have positive probability of having an $L \times N$-barrier inside the halfbox favorable for the type of particle that occupied this point. On the other hand we also have positive probability, independent of N, that there is non mark of infection in the regions $\tilde{A}_1 \times [\tilde{c}_k + \tilde{S} - 1, \tilde{c}_k + \tilde{S}]$, $\tilde{A}_3 \times [\tilde{c}_k + \tilde{S} - 1, \tilde{c}_k + \tilde{S}]$ and $\tilde{A}_5 \times [\tilde{c}_k + \tilde{S} - 1, \tilde{c}_k + \tilde{S}]$ and there is non particle alive at this regions at time $\tilde{c}_k + \tilde{S}$. Summing up: outside an event with exponential small probability if $\min\{\tau_N^{L,\zeta}, \tau_N^{L,\mathbf{1,2}}\} > \tilde{c}_N + \tilde{S}$ we can find a time $\tilde{c}_k$ such that

$(z_k^\zeta, \tilde{c}_k)$ and $(z_k^{\mathbf{1,2}}, \tilde{c}_k)$ are $L \times N$-barriers inside $[-L, L] \times [1, N]$,

$(\mathsf{z}_k^{\mathbf{1,2}}, \tilde{c}_k)$ and $(\mathsf{z}_k^{\mathbf{1,2}}, \tilde{c}_k)$ are $L \times N$-barriers inside $[-L, L] \times [-N+1, 0]$,

there is non particle alive at time $\tilde{c}_k + \tilde{S}$ in $\tilde{A}_1$, $\tilde{A}_3$ and $\tilde{A}_5$.

Therefore, in this event $\zeta_L^{\zeta,N}(\tilde{c}_N + \tilde{S}) = \zeta_L^{\mathbf{1,2},N}(\tilde{c}_N + \tilde{S})$. For $\tilde{e}_N = \tilde{c}_N + \tilde{S}$ and $\tilde{d}_N = \tilde{b}_N$ we can state the following proposition

Proposition 3.4. *There are sequences $\tilde{e}_N$ and $\tilde{d}_N$ that satisfy*

(i) $\lim_{n\to\infty} \inf_{\zeta\in\mathcal{C}_L} \mathbb{P}^{\zeta}(\zeta_L^{\mathbf{1},\mathbf{2},N}(\tilde{e}_N) = \zeta_L^N(\tilde{e}_N)$ *or* $\tau_N^L < \tilde{e}_N) = 1$;

(ii) $\frac{\tilde{d}_N}{\tilde{e}_N} \to \infty$;

iii) $\lim_{n\to\infty} \mathbb{P}^{\mathbf{1},\mathbf{2}}(\tau_N^L < \tilde{d}_N) = 0$,

where $\mathcal{C}_L = \{\zeta \in \{0,1,2\}^{[-L,L]\times[-N+1,N]} : \exists \ x, y \ \ \zeta(x) = 1, \zeta(y) = 2\}$.

Let β_N^L such that $\mathbb{P}(\tau_N^{L,\mathbf{1},\mathbf{2}} > \beta_N^L) = e^{-1}$. As we prove in Section 2.3.1 Proposition 3.4 imply

$$\lim_{N\to\infty} \left| \mathbb{P}(\tau_N^{L,\mathbf{1},\mathbf{2}} > \beta_N^L(t+s)) - \mathbb{P}(\tau_N^{L,\mathbf{1},\mathbf{2}} > \beta_N^L t)\mathbb{P}(\tau_N^{L,\mathbf{1},\mathbf{2}} > \beta_N^L s) \right| = 0,$$

for every $t \geq 0$ and we obtain the convergence in distribution of $\{\tau_N^{L,\mathbf{1},\mathbf{2}}/\beta_N^L\}_N$ to the exponential distribution with mean 1.

Chapter 4

More results for the one-dimensional contact process with two types of particles and priority

During this chapter we are dealing with the contact process in dimension 1. The first process that we study is the contact process in infinite volume with two types of particles and priority, with initial configuration $\mathbb{1}_{(-\infty,0]} + 2\mathbb{1}_{[1,\infty)}$. In this case we consider the range $R \geq 1$. For this process we prove the tightness of the positive part of the rightmost particle 1 at time t, that we denote by

$$r_t^1 = \max\{x : \zeta_t^{\mathbf{1},\mathbf{2}}(x) = 1\}. \tag{4.0.1}$$

Having the tightness we prove the existence of an invariant measure which gives measure 1 for the configurations in $\{0,1,2\}^{\mathbb{Z}}$ with infinite particles of type 1 and type 2. The statements are as follows

Theorem 4.1. *For every $\epsilon > 0$ there exists M such that*

$$\mathbb{P}(\max\{r_t^1, 0\} \leq M) \geq 1 - \epsilon \tag{4.0.2}$$

for every $t > 0$.

Theorem 4.2. *For the contact process with two types of particles there exists an invariant measure* ν *that satisfies*

$$\nu(\zeta \in \{0,1,2\}^{\mathbb{Z}} : |\{x : \zeta(x) = 1\}| = \infty, |\{x : \zeta(x) = 2\}| = \infty) = 1.$$

The other process that we study is the contact process with two types of particles in the interval $[-N+1, N]$ and initial configuration $\mathbb{1}_{[1,N]} + 2\mathbb{1}_{[-N+1,0]}$. We denote this process by $\zeta^{\mathbf{2},\mathbf{1},N}(t)$. Different from what we have seen so far, at time 0 the particles of type 1 are in the region favorable to the particles of type 2 and vice-versa. We state that, in the nearest neighbor case, the time when one of the families dies in the interval $[-N+1, N]$, $\tau_N^{\mathbf{2},\mathbf{1}}$, is at most linear with respect to N, when N tends to infinite. More precisely in Section 4.3 we prove the following theorem

Theorem 4.3. *Let* $R = 1$ *and* α *as in* (1.1.9)*, then*

$$\lim_{N \to \infty} \frac{\tau_N^{\mathbf{2},\mathbf{1}}}{N} = \frac{1}{\alpha} \text{ in Probability.} \tag{4.0.3}$$

4.1 Graphical construction of the contact process with two types of particles and priority in $\mathbb{Z}$

In Section 1.2 we give a definition of the contact process with two types of particles and priority restricted to a finite interval, $[-N+1, N]$. In this chapter we deal with the version of this process in $\mathbb{Z}$. For this process we also have a definition using the Harris construction.

Let A and B be two disjoint subsets of $\mathbb{Z}$. We denote the contact process with two types of particles and priority with initial configuration $\mathbb{1}_A + 2\mathbb{1}_B$ as $\zeta_t^{A,B}$. We will define the process almost surely for every time $q \in \mathbb{Q}^+$, the extension for all times we defined as $\zeta_t^{A,B}(x) = \lim_{q \to t^+} \zeta_q^{A,B}(x)$, almost surely. In this way, we also guarantee that $\zeta_t^{A,B}$ is a càdlàg stochastic process. First, we observe that $\mathbb{P}(|\tilde{\xi}^0(t)| < \infty) = 1$ for every $q \in \mathbb{Q}^+$ because $|\tilde{\xi}^0(q)| \le RN_q$, with $\{N_t\}_t$ a Poisson process with rate 2λ. Then for every $x \in \mathbb{Z}$ and every $q \in \mathbb{Q}^+$, $|\tilde{\xi}^x(q)| < \infty$ almost surely. Therefore, by the definition provided in Section 1.2, for an N such that $\tilde{\xi}^x(q) \subset [-N+1, N]$

$$\zeta_q^{A\cap[-N+1,N],B\cap[-N+1,N],N}(x) = \zeta_q^{A\cap[-M+1,M],B\cap[-M+1,M],M}(x)$$

for every $M \geq N$.

Hence, we define $\zeta_q^{A,B}(x) = \zeta_q^{A\cap[-N+1,N],B\cap[-N+1,N],N}(x)$ in $\Omega' = \{|\tilde{\xi}^x(q)| < \infty, \forall x \in \mathbb{Z}, \forall q \in \mathbb{Q}^+\}$, which has probability 1.

4.2 One-dimensional contact process with two types of particles in $\mathbb{Z}$

In this section we present some results for the contact process with two types of particles and priority in $\mathbb{Z}$ with initial configuration $\mathbb{1}_{(-\infty,0]} + 2\mathbb{1}_{[1,\infty)}$ and range $R > 1$. In Subsection 4.2.1 we present some results for the k-dependent percolation systems with closure close to 0. We dedicate Subsection 4.2.2 to the proofs of Theorem 4.1 and Theorem 4.2. First we apply the results of Section 4.2.1 to the Mountford-Sweet renormalization and we obtain Theorem 4.1. Theorem 4.2 is a consequence of Theorem 4.1.

4.2.1 More results on k-dependent percolation system with closure close to 0

During this subsection we use the notation introduced in Section 1.3 for the k-dependent percolation system. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a k-dependent oriented percolation system with small closure. We define the following events:

$$A_n = \{\exists\, y : y > n/2 \text{ and } (2,0) \rightsquigarrow (y,n)\},$$

$$\Gamma_n = \left\{ \begin{array}{c} \text{there exists a path connecting } \{2\} \times \{0\} \rightsquigarrow \mathbb{Z} \times \{n\} \text{ such that} \\ \text{this path does not intersect the set} \{(m,s) \in \Lambda : m \leq s/2\} \end{array} \right\}$$

and

$$\Gamma = \bigcap_{n \in \mathbb{N}} \Gamma_n.$$

The next lemma will be used in the proof of Theorem 4.1.

Lemma 4.1. *For $\epsilon > 0$ and $k \in \mathbb{N}$ there exist p_0 and δ such that*

i) $\hat{\mathbb{P}}_p(\exists\ n : A_n = \emptyset) < \epsilon$ *for all* $p \in [p_0, 1]$*;*

ii) $\hat{\mathbb{P}}_p(\Gamma) > 1 - \epsilon$ *for all* $p \in [p_0, 1]$*;*

iii) if $(\Omega, \mathcal{F}, \mathbb{P})$ *is a k-dependent oriented percolation system with closure below* δ*, then*

$$\mathbb{P}\left(\left\{\begin{array}{c}\exists \text{ a path connecting } [n/2, +\infty) \times \{0\} \cap \Lambda \rightsquigarrow (2, n) \\ \text{and this path does not intersects the set} \\ \{(m, s) \in \Lambda : m \leq n/2 - s/2\}\end{array}\right\}\right) > 1 - \epsilon,$$

for all n*.*

Proof. Observe that in the set $\{|C_{(0,2)}| = \infty\}$

$$\begin{aligned}\{x : (2, 0) \rightsquigarrow (x, n)\} &\cap [\hat{l}_n^2, \hat{r}_n^2] \\ &= \{x : \exists\, y \in (-\infty, 0] \text{ such that } (y, 0) \rightsquigarrow (x, n)\} \cap [\hat{l}_n^2, \hat{r}_n^2].\end{aligned}$$

Therefore, we conclude that

$$\hat{r}_n = \hat{r}_n^{\{2\}} \text{ a.s in } \{|C_{(2,0)}| = \infty\}. \tag{4.2.1}$$

The proof of item (i) follows by (1.3.4), (1.3.5), (4.2.1) and the following inequality

$$\hat{\mathbb{P}}_p(\exists\ n : A_n = \emptyset)) \leq \hat{\mathbb{P}}_p(C_{(2,0)} \text{ is finite}) + \hat{\mathbb{P}}_p(\exists\ n \geq 1 : \hat{r}_n < n/2).$$

To prove item (ii), we first observe that by the definition of the events A_n we have that

$$\Gamma^c \subset \bigcup_{n \geq 0} A_n.$$

Item (i) implies that there exists p_0 such that for all $p \in (p_0, 1]$

$$\mathbb{P}_p(\Gamma^c) \leq \mathbb{P}_p(\exists\ n : A_n = \emptyset) < \epsilon.$$

To obtain item (*iii*), we first observe that the event

$$A=\left\{\begin{array}{c}\exists \text{ a path connecting } [n/2,+\infty)\times\{0\}\cap\Lambda \rightsquigarrow (2,n)\\ \text{and this path does not intersects the set}\\ \{(m,s)\in\Lambda : m\leq n/2-s/2\}\end{array}\right\},$$

is decreasing. From Theorem 1.3, there exists $\delta>0$ such that for all k-dependent oriented percolation system $(\Omega,\mathcal{F},\mathbb{P})$ with closure under δ, we have that

$$\begin{aligned}&\mathbb{P}\left(\left\{\begin{array}{c}\exists \text{ a path connecting } [n/2+2,+\infty)\times\{0\}\cap\Lambda \rightsquigarrow (2,n)\\ \text{and this path does not intersects the set}\\ \{(m,s)\in\Lambda : m\leq (n+4)/2-s/2\}\end{array}\right\}^c\right)\\
&\leq \mathbb{P}_{p_0}\left(\left\{\begin{array}{c}\exists \text{ a path connecting } [n/2+2,+\infty)\times\{0\}\cap\Lambda \rightsquigarrow (2,n)\\ \text{and this path does not intersects the set}\\ \{(m,s)\in\Lambda : m\leq n/2-s/2\}\end{array}\right\}^c\right)\\
&= \mathbb{P}_{p_0}\left(\left\{\begin{array}{c}\exists \text{ a path connecting } \{2\}\times\{0\} \rightsquigarrow \mathbb{Z}\times\{n\} \text{ such that}\\ \text{this path does not intersect the set}\\ \{(m,s)\in\Lambda : m\leq s/2\}\end{array}\right\}^c\right)\\
&= \mathbb{P}_{p_0}((\Gamma_n)^c)\leq \mathbb{P}_{p_0}((\Gamma)^c)\leq\epsilon,\end{aligned}$$

where the second equality above is exclusively true for the Bernoulli product measure. □

4.2.2 Proofs of Theorem 4.1 and Theorem 4.2

Proof of Theorem 4.1: Using Lemma 4.1 and Proposition 1.1 for $\epsilon>0$ there exist k, $\hat{K}$ and $\hat{N}$ such that Ψ is a k-dependent percolation system and

$$\mathbb{P}\left(\left\{\begin{array}{c}\exists \text{ a path connecting } [2+n/2,+\infty)\times\{0\}\cap\Lambda \rightsquigarrow (2,n)\\ \text{and this path does not intersects the set}\\ \{(m,s)\in\Lambda : m\leq (n+4)/2-s/2\}\end{array}\right\}\right)>1-\epsilon$$

for all n. In fact, we need a translation of the event in the probability above. We denote the unit of translation by $\jmath$ and define it as follows

$$\jmath = \begin{cases} \left\lceil 2\alpha\hat{K}\hat{N} \right\rceil & \text{if } n \text{ is } \textit{even}, \\ \left\lceil 2\alpha\hat{K}\hat{N}+1 \right\rceil & \text{if } n \text{ is } \textit{odd}. \end{cases} \tag{4.2.2}$$

By the invariance translation of Harris construction, we have that

$$\mathbb{P}\left(\left\{ \begin{array}{c} \exists \text{ a path connecting } [n/2 - s/2 + \jmath, +\infty) \times \{0\} \cap \Lambda \rightsquigarrow (\jmath + 2, n) \\ \text{and this path does not intersects the set} \\ \{(m,s) \in \Lambda : m \leq n/2 - s/2 + \jmath\} \end{array} \right\}\right) > 1 - \epsilon.$$

We take $t \geq \hat{K}\hat{N}$ and choose $n = \left\lfloor \frac{t}{\hat{K}\hat{N}} \right\rfloor + 1$. For this n, with probability larger than $1-\epsilon$, there exists a sequence $\{m_k\}_{0 \leq k \leq n}$ such that

$$\begin{aligned} &\Psi(m_k, k) = 1 \ \forall \ k \ \in \{0, \dots, n\}, \\ &|m_{k+1} - m_k| = 1, \\ &m_k \geq n/2 - k/2 + \jmath. \end{aligned}$$

Also, we denote

$$B_n = \cup_{0 \leq k \leq n} I^{\hat{K},\hat{N}}_{(m_k,k)} \cup J^{\hat{K},\hat{N}}_{(m_k,k)}.$$

By the properties (1.4.2) and (1.4.3) of the Mountford-Sweet renormalization, we observe that in the trajectory of the contact process $t \mapsto \xi(t)(\mathcal{H})$ every infected site in B_n descends from $I^{\hat{K},\hat{N}}_{(m_0,0)} \subset \mathbb{Z}^+$. Property (1.4.4) of the renormalization and the fact that $m_k \geq n/2 - k/2 + \jmath$ imply that such sites are infected for paths to the right of $\{\frac{(\jmath+2)N}{2} - 2\alpha\hat{K}\hat{N}\} \times [0, +\infty) \subset \mathbb{Z}^+ \times [0, +\infty)$.

In this situation, for any $y \in \xi(t) \cap [\frac{(\jmath+2)N}{2} + 2\alpha\hat{K}\hat{N}, +\infty)$ we have two possibilities: $\mathbb{Z} \times \{0\} \to (y,t)$ with a path that intersects B_n or for a path that stays for all times to the right of B_n. In both cases, we can construct a path contained in $\{\frac{(\jmath+2)N}{2} - 2\alpha\hat{K}\hat{N}\} \times [0, +\infty)$. By Lemma 1.1, $\zeta^{\mathbf{1},\mathbf{2}}_t(y) = 2$. We can conclude that

$$1 - \epsilon \ \leq \mathbb{P}\left((r^1)^+_t < \frac{(\jmath+2)N}{2} + 2\alpha\hat{K}\hat{N} \right),$$

for all $t \geq \hat{K}\hat{N}$. Then, we choose $M_1 = 2\alpha\hat{K}\hat{N} + \jmath N/2$ which does not depend on t. In the

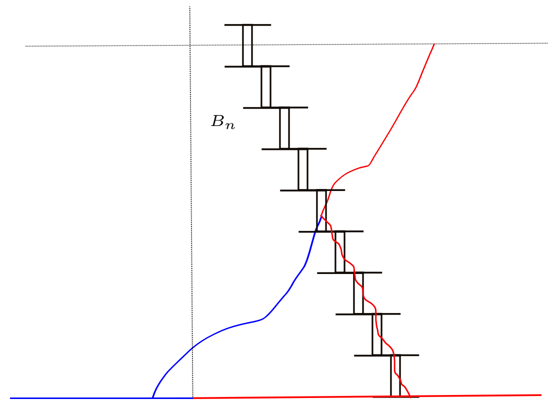

Figure 4.1: We represent the blue paths referent to particles of type 1 and the red ones to particles of type 2.

case $t < \hat{K}\hat{N}$, observe that

$$\sup_{0 \le t \le KN} (r_t^1)^+ < +\infty,$$

then there exists M_2 such that

$$1 - \epsilon < \mathbb{P}((r_t^1)^+ < M_2),$$

for all $0 \le t \le \hat{K}\hat{N}$. $\square$

Proof of Theorem 4.2: At the space $\{0,1,2\}^{\mathbb{Z}}$ we define the following metric

$$\tilde{\rho}(\eta, \eta') = \sum_{x \in \mathbb{Z}} \frac{|\eta(x) - \eta'(x)|}{2^{|x|}(1 + |\eta(x) - \eta'(x)|)},$$

where $\eta, \eta' \in \{0,1,2\}^{\mathbb{Z}}$. Observe that for all η, η', $\tilde{\rho}(\eta, \eta') \le 2$, also with this metric $\{0,1,2\}^{\mathbb{Z}}$ is a complete space. Then $(\{0,1,2\}^{\mathbb{Z}}, \tilde{\rho})$ is a compact space. We denote ν_t as the law of $\zeta_t^{\mathbf{1,2}}$ and

$$\mu_T(A) = \frac{1}{T} \int_0^T \nu_t(A) dt,$$

for all Borel set A.

Because the space is compact, $\{\mu_T\}_T$ is a tight family. Let $\{\mu_{T_k}\}_k$ be a convergent subsequence to a measure ν. Using Proposition 1.8 and Chapter I of [12], we have that ν is an invariant measure for the process. Let b_n be such that b_n converges to infinity and define the following sets

$$\mathcal{A}_{b_n} = \left\{ \zeta \in \{0,1,2\}^{\mathbb{Z}} : \frac{|\{x : \zeta(x) = 1\} \cap [-b_n, -M]|}{b_n - M} \geq \frac{\rho}{2}, \frac{|\{y : \zeta(y) = 2\} \cap [M, b_n]|}{b_n - M} \geq \frac{\rho}{2} \right\}$$

and

$$\mathcal{B}_{b_n} = \left\{ \xi \in \{0,1\}^{\mathbb{Z}} : \frac{|\{x : \xi(x) = 1\} \cap [-b_n, -M]|}{b_n - M} \geq \frac{\rho}{2}, \frac{|\{y : \xi(y) = 1\} \cap [M, b_n]|}{b_n - M} \geq \frac{\rho}{2} \right\},$$

where $\rho = \mathbb{P}(T^0 = \infty)$. From Theorem 4.1 we know that for $\epsilon > 0$ exists M such that for all $t \geq 0$

$$\mathbb{P}(r_t^+ > M) + \mathbb{P}(l_t^- > M) \leq \epsilon.$$

We observe that for all $t \geq 0$

$$\nu_t(\mathcal{A}_{b_n}^c) \leq \mathbb{P}(\xi^{\mathbb{Z}}(t) \in \mathcal{B}_{b_n}^c) + \mathbb{P}(r_t^+ > M) + \mathbb{P}(l_t^- > M) \leq \mu(\mathcal{B}_{b_n}^c) + \epsilon,$$

where μ is the non-trivial invariant measure of the classic contact process. Then,

$$\mu_{T_k}(\mathcal{A}_{b_n}^c) \leq \mu(\mathcal{B}_{b_n}^c) + \epsilon.$$

By the convergence of the sequence $\{\mu_{T_k}\}_k$ to μ, we have that

$$\nu(\mathcal{A}_{b_n}^c) \leq \mu(\mathcal{B}_{b_n}^c) + \epsilon.$$

As a consequence of the ergodicity of μ, we have that

$$\lim_{n\to\infty} \nu(\mathcal{A}_{b_n}^c) \leq \lim_{n\to\infty} \mu(\mathcal{B}_{b_n}^c) + \epsilon = \epsilon.$$

Observe that $S = \limsup A_{b_n}$ is a subset of $\{\zeta \in \{0,1,2\}^{\mathbb{Z}} : |x : \zeta(x) = 1| = \infty \text{ and } |x : \zeta(x) = 2| = \infty\}$. Then, we conclude that

$$\nu(\{\zeta \in \{0,1,2\}^{\mathbb{Z}} : |x : \zeta(x) = 1| = \infty \text{ and } |x : \zeta(x) = 2| = \infty\}) = 1,$$

which proves Theorem 4.2. □

4.3 Proof of Theorem 4.3

As we mention in the beginning of this chapter, in this section we are dealing with the contact process with two types of particles and priority with range $R = 1$. The priority will be the same as before: particles of type 1 have priority in $(-\infty, 0]$ and particles of type 2 in $[1, \infty)$. But now we explore the behavior of the process when the initial configuration is $2\mathbb{1}_{(-\infty,0]} + \mathbb{1}_{[1,\infty)}$, that is, in the initial configuration the particles of type 1 are in the region favorable to the particles of type 2 and vice-versa.

We denote by $\zeta_t^{\mathbf{2,1}}$ the contact process with two types of particles and the particles of type 1 having priority in $(-\infty, 0]$ and the particles of type 2 in $[1, \infty)$, with initial configuration $2\mathbb{1}_{(-\infty,0]} + \mathbb{1}_{[1,\infty)}$. Also, we use the notation $\zeta_{s,t}^{\mathbf{2,1}}$ for the contact process with two types of particles with regions of priority and initial configuration as before but constructed in $\Theta_{(0,s)}(\mathcal{H})$, the translation of Harris graph to time s. We denoted by $\zeta^{\mathbf{2,1},N}(t)$ the contact process with two types of particles and priority restricted to the interval $[-N+1, N]$ with initial configuration $2\mathbb{1}_{[-N+1,0]} + \mathbb{1}_{[1,N]}$.

We define $\hat{\mathcal{C}}$ as the set of configurations $\zeta_0 \in \{0, 1, 2\}^{\mathbb{Z}}$ such that the sites occupied by particles of type 2 are at the left of the sites occupied by particles of type 1. We observe that, since we are in the nearest neighbour scenario, the set $\hat{\mathcal{C}}$ is invariant for the contact process with two types of particles and priority. Let ζ_0 be a configuration in $\hat{\mathcal{C}}$, we define the leftmost 1 and the rightmost 2 at time t as follows

$$\hat{l}_t^{\zeta_0} = \min\{x : \zeta_t^{\zeta_0}(x) = 1\}$$

and

$$\hat{r}_t^{\zeta_0} = \max\{x : \zeta_t^{\zeta_0}(x) = 2\}.$$

In the special case $\zeta_0 = 2\mathbb{1}_{(-\infty,0]} + \mathbb{1}_{[1,\infty)}$, we write $\hat{r}_t^1$ and $\hat{l}_t^2$. For the classic contact process, we reserve the usual notation $r_t^{(-\infty,0]}$ for the rightmost occupied site at time t and $l_t^{[0,+\infty)}$ for the leftmost occupied site at time t, which were defined in (1.1.7) and (1.1.8) respectively.

For $A \subset \mathbb{N}$, define χ_t^A as follows

$$\chi_t^A = \{x: \text{ there is a path inside } \mathbb{N} \text{ from } (y,0) \text{ to } (x,t) \text{ for some } y \in A\},$$

and denote ρ^+ as

$$\rho^+ = \mathbb{P}(\chi_t^0 \neq \emptyset, \forall\, t).$$

We also denote by λ_c^+ the critical parameter for the contact process in $\mathbb{N}$. In Corollary 2.5 of [2] was proved that $\lambda_c^+ = \lambda_c(\mathbb{Z})$. Since we are dealing with the supercritical contact process in $\mathbb{Z}$, the process restricted to $\mathbb{N}$ is also supercritical. In the supercritical case the process $\chi_t^{\mathbb{N}}$ converge in distribution, when t goes to infinity, to a non trivial invariant probability measure that we denote by μ_+. Another useful observation is the fact that $\chi_t^{\mathbb{N}}$ is stochastically larger than μ_+, and this is obvious by the property of attractiveness.

In order to obtain Theorem 4.3, we first give some results for the process defined in infinite volume $\zeta_t^{\mathbf{2},\mathbf{1}}$. For ζ_0 a configuration in $\hat{\mathcal{C}}$ we define the following stopping times:

$$S_0^{\zeta_0} = 0;\; S_1^{\zeta_0} = \inf\{t > 0; \hat{r}_t^{\zeta_0} > 0 \text{ or } \hat{l}_t^{\zeta_0} \leq 0\}.$$

Being defined $S_0^{\zeta_0}, S_1^{\zeta_0}, \ldots, S_{k-1}^{\zeta_0}$, the k-th stopping time is given by

$$S_k^{\zeta_0} = \begin{cases} \inf\{t > S_{k-1} : \hat{l}_t^{\zeta_0} \leq 0\} & \text{if } \hat{r}_{S_{k-1}}^{\zeta_0} > 0, \\ \inf\{t > S_{k-1} : \hat{r}_t^{\zeta_0} > 0\} & \text{if } \hat{l}_{S_{k-1}}^{\zeta_0} \leq 0. \end{cases}$$

These are the times when in the process $\{\zeta_t^{\zeta_0}\}_t$ there is a cross from a particle of type 1 to $(-\infty, 0]$ or a particle of type 2 to $[1, \infty)$. In the special case $\zeta_0 = 2\mathbb{1}_{(-\infty,0]} + \mathbb{1}_{[1,\infty)}$, we omit the superscript. The following variable counts the number of finite stopping times S_k

$$G = k \text{ in } \{S_k < +\infty\} \cap \{S_{k+1} = +\infty\} \text{ for } k \geq 0. \tag{4.3.1}$$

In the next lemma we prove that the variable G is stochastically dominated by a geometric distribution.

Lemma 4.2. *For G as in* (4.3.1) *we have that*

$$\mathbb{P}(G \geq k) \leq (1 - \rho^+)^{k-1},\; k \geq 1. \tag{4.3.2}$$

Furthermore, for $k = 0$ we have that

$$\mathbb{P}(G = 0) = 0. \tag{4.3.3}$$

Proof. To obtain (4.3.2), we first observe that for any $\zeta_0 \in \hat{\mathcal{C}}$ the processes $\zeta_t^{\zeta_0}$ and χ_t^1 are defined using the same Harris graph, therefore we have a coupling between both processes. For a configuration ζ_0 such that $\zeta_0(1) = 2$, if $\chi_t^1 \neq \emptyset$ for all t, then for the process $\{\zeta_t^{\zeta_0}\}_t$ there is non cross from a particle of type 1 to $(-\infty, 0]$, because the particles of type 2 have the priority in $[1, \infty)$. From this argument follows the next inequality

$$\mathbb{P}(\zeta_\cdot^{\zeta_0} \in \{S_1^{\zeta_0} < \infty\}) \leq \mathbb{P}(\exists\, t \geq 0 : \chi_t^1 = \emptyset) = 1 - \rho^+. \tag{4.3.4}$$

In the case $\zeta_0(0) = 1$, we also have (4.3.4) by the symmetry of Harris construction. Since $\zeta^{\mathbf{2,1}}_{S_{k-1}}$ has a particle of type 2 in the site 1 or a particle of type 1 in the site 0, by the Strong Markov property and (4.3.4) we have that

$$\mathbb{P}(G \geq k) = \mathbb{P}(\zeta^{\mathbf{2,1}}_{S_{k-1},\cdot} \in \{S_1^{\zeta^{\mathbf{2,1}}_{S_{k-1}}} < \infty\}; S_{k-1} < \infty) \leq (1 - \rho^+)\mathbb{P}(S_{k-1} < \infty)$$

and by induction on k, for all $k \geq 1$ we obtain (4.3.2). To obtain (4.3.3), we first define the next stopping time

$$T = \inf\{t : (-\infty, 0] \times \{0\} \to (1, t) \text{ or } [1, \infty) \times \{0\} \to (0, t)\}.$$

The fist step will be to prove the following inclusion

$$\{T < \infty\} \subset \{G > 0\}. \tag{4.3.5}$$

Fix a realization in the set on the left member of (4.3.5) such that

$$(-\infty, 0] \times \{0\} \to (1, T).$$

In this case, by the definition of T there is non path in the Harris graph connecting $[1, \infty) \times \{0\}$ with $(-\infty, 0] \times [0, T)$. Therefore, for the contact process with two types of particles and priority with initial configuration $2\mathbb{1}_{(-\infty,0]} + \mathbb{1}_{[1,\infty)}$ there is non particle of type 1 in $(-\infty, 0] \times [0, T)$. For this reason, the path connecting $(-\infty, 0] \times \{0\}$ with $(1, T)$ is a path of particles of type 2 that ends in a site in which type 2 has priority, so $\zeta_T^{\mathbf{2},\mathbf{1}}(1) = 2$. Then in this case $G > 0$. The argument for the case $[1, \infty) \times \{0\} \to (0, T)$ is very similar. Hence, we have proved the inclusion (4.3.5). Equation (4.3.3) follows from (4.3.5) and the fact that the event $\{T = 0\}$ has probability equal to zero. □

The previous lemma implies that the variable G is finite and more than that, it is stochastically dominated by a geometric distribution. By the nature of the dynamics, after S_G only one type of particle has the priority in all $\mathbb{Z}$. Therefore, after this time, the process behaves as the Grass-Bushes-Trees (G-B-T) process.

Let us recall the definition of the G-B-T model. Let A and B be two disjoint subsets of $\mathbb{Z}$, we denote by $\tilde{\zeta}_t^{A,B}$ the G-B-T process where the particles of type 2 (the trees) have the priority and with initial configuration $\mathbb{1}_A + 2\mathbb{1}_B$. Given the Harris graph, this process is defined as follows

$$\tilde{\zeta}_t^{A,B}(x) = \begin{cases} 2, & \text{if } B \times \{0\} \to (x,t) \\ 1, & \text{if } B \times \{0\} \nrightarrow (x,t) \text{ and } A \times \{0\} \to (x,t) \\ 0, & \text{other case,} \end{cases}$$

for $x \in \mathbb{Z}$ and $t \geq 0$. In the especial case $A = [1, \infty)$ and $B = (-\infty, 0]$, we use the notation $\tilde{\zeta}_t^{\mathbf{2},\mathbf{1}}$. We will refer to $\tilde{\zeta}_t^{A,B,N}$ for the G-B-T process restricted to $[-N+1, N]$ with initial configuration $\mathbb{1}_A + 2\mathbb{1}_B$, where A and B are disjoint subsets of $[-N+1, N]$. For $A = [1, N]$ and $B = [-N+1, 0]$, we use the notation $\tilde{\zeta}_t^{\mathbf{2},\mathbf{1},N}$.

We state the fact that we can compare the contact process with two types of particles and priority with the G-B-T process in the following remark.

Remark 4.1. *Let A and B be disjoint subsets of $\mathbb{Z}$ such that $A \subset [0, \infty)$, $B \subset (-\infty, 1]$ and $1 \in B$, then*

$$\{S_2^{A,B} = +\infty\} \subset \{\tilde{\zeta}_t^{A,B} = \zeta_t^{A,B}\, \forall\, t \geq 0\}.$$

Once we have the comparison between the contact process with two types of particles and priority with the G-B-T model, we need another important ingredient to prove Theorem 4.3, wich we state in the next lemma. In this lemma we prove that for the G-B-T model restricted to $[-N+1, N]$ and N large enough, with probability close to one, the extinction time of the bushes is at most linear in N if in the initial configuration there are a large (on N) number of trees. Before proving this result let us define the time of extinction of the bushes. For A and B two disjoint subsets of $\mathbb{Z}$ we define the time when the bushes or the trees become extinct for the G-B-T model restricted to the interval $[-N+1, N]$ as

$$\tilde{\tau}_N^{A,B} = \inf\{t > 0 : \{x : \tilde{\zeta}_t^{A,B,N}(x) = 1\} = \emptyset \text{ or } \{x : \tilde{\zeta}_t^{A,B,N}(x) = 2\} = \emptyset\}.$$

Lemma 4.3. *Let $0 < \eta < 1$, $0 < \rho < 1$, A and B be subsets of $[-N+1, N]$ such that $B \subset [-N+1, 1]$ and $A \subset [2, N]$. Assume also that $|B \cap [-\eta N, 1]| > \rho\eta N/2$. In this conditions and for δ such that $0 < \delta < \alpha$ we have that*

$$\lim_{N\to\infty} \mathbb{P}(\tilde{\tau}_N^{A,B} > N\beta) = 0,$$

where $\beta = (1+\eta)/(\alpha - \delta)$.

Proof. To simplify the notation let us denote $B \cap [-\eta N, 0]$ by B_N. Observe that

$$\begin{aligned} \{\tilde{\tau}_N^{A,B} > N\beta\} = \{&\tilde{\tau}_N^{A,B} > N\beta; T^{B_N} > N\beta\} \\ &\cup \{\tilde{\tau}_N^{A,B} > N\beta; T^{B_N} \leq N\beta\}. \end{aligned} \tag{4.3.6}$$

The idea is to estimate the probability of each event in the right member of (4.3.6). For the probability of the second event, we use a result proved in Section 10 of [6], which states that for every subset D of $\mathbb{Z}$

$$\mathbb{P}(T^D < \infty) \leq e^{-c|D|}, \tag{4.3.7}$$

where c is a positive constant. Hence, the probability of the second term is less than $e^{-\frac{c\rho\eta N}{2}}$. Now, we focus on the first term of (4.3.6). The first step will be to prove

$$\lim_{N\to\infty} \mathbb{P}(\tilde{\tau}_N^{A,B} > N\beta; T^{B_N} > N\beta; T^{B_N}_{[-N+1,N]} \leq N\beta) = 0. \tag{4.3.8}$$

Observe that

$$\begin{aligned} &\{\tilde{\tau}_N^{A,B} > N\beta; T^{B_N} > N\beta\} \\ &\qquad \subset \{\tilde{\tau}_N^{A,B} > N\beta; T^{B_N} > N\beta; l_{N\beta}^{B_N} < -(\alpha-\delta)\beta N; r_{N\beta}^{B_N} > (\alpha-\delta)\beta N\} \\ &\qquad \cup \{\tilde{\tau}_N^{A,B} > N\beta; T^{B_N} > N\beta; l_{N\beta}^{B_N} \geq -(\alpha-\delta)\beta N \text{ or } r_{N\beta}^{B_N} \leq (\alpha-\delta)\beta N\}. \end{aligned} \tag{4.3.9}$$

By (1.1.9), the limit when N goes to infinity of the probability of the second term in the right member of (4.3.9) is zero. In the first term, by the fact that $l_{N\beta}^{B_N}$ is less than $-N+1$ and $r_{N\beta}^{B_N}$ is greater than N and the path crossing property, we have that $\xi_{[-N+1,N]}^{B_N}(N\beta) = \xi_{[-N+1,N]}^{B}(N\beta)$. Since also in this event $T_{[-N+1,N]}^{B} > N\beta$, then in the first term of (4.3.9) we have $T_{[-N+1,N]}^{B_N} > N\beta$. These arguments yield (4.3.8). Thus, instead of proving that the probability of the first term in the right member of (4.3.6) goes to zero when N goes to infinity we prove that

$$\lim_{N\to\infty} \mathbb{P}(\tilde{\tau}_N^{A,B} > N\beta; T_{[-N+1,N]}^{B_N} > N\beta) = 0. \tag{4.3.10}$$

For this purpose observe that

$$\begin{aligned}
&\{\tilde{\tau}_N^{A,B} > N\beta; T_{[-N+1,N]}^{B_N} > N\beta\} \\
&\subset \{\tilde{\tau}^{A,B} > N\beta; \exists\, y \in B_N : T_N^y > N\beta \text{ and } \xi_{[-N+1,N]}^{\mathbb{1}}(N\beta) = \xi_{[-N+1,N]}^{y}(N\beta)\} \\
&\cup \{\tilde{\tau}^{A,B} > N\beta; \exists\, y \in B_N : T_N^y > N\beta \text{ and } \xi_{[-N+1,N]}^{\mathbb{1}}(N\beta) \neq \xi_{[-N+1,N]}^{y}(N\beta)\}.
\end{aligned}$$

The first event above is empty by the priority of the particles of type 2 in all $[-N+1, N]$. The probability of the second event goes to zero as N goes to infinity because

$$\{T_N^y > N\beta \text{ and } \xi_{[-N+1,N]}^{\mathbb{1}}(N\beta) \neq \xi_{[-N+1,N]}^{y}(N\beta)\} \subset \{l_{\beta N}^{[0,\infty)} > -N+1 \text{ or } r_{\beta N}^{(-\infty,-\eta N]} < N\}$$

and by (1.1.9) the probability of this last event goes to zero when N goes to infinity. □

Remark 4.2. *Using the path crossing property and the attractiveness of the contact process it is possible to prove that there exists a sequence β_N that goes to zero such that for all $s > 0$*

$$\mathbb{P}(T_{[-N,N]}^{\mathbb{1}} > s; \xi_{[-N,N]}^{\mathbb{1}}(s) \notin \mathcal{B}_{bN}) \leq \beta_N,$$

where $0 < b < 1$ and for $\rho = \mu(\xi : \xi(0) = 1)$

$$\mathcal{B}_{bN} = \left\{ \xi \in \{0,1\}^{[-N,N]} : \frac{|\xi \cap [-bN, 1]|}{bN} \geq \frac{\rho}{2} \right\}.$$

For details on this result see the proof of Theorem 4.20, page 257 in [17]. Then, for the *stopping time S_k we have that*

$$\lim_{N\to\infty}\mathbb{P}(T^{\mathbb{1}}_{[-N,N]} > S_k; \xi^{\mathbb{1}}_{[-N,N]}(S_k) \notin \mathcal{B}_{bN}) = 0. \tag{4.3.11}$$

Now we are finally ready to prove Theorem 4.3

Proof of Theorem 4.3. For $\delta > 0$ and $0 < \gamma < 1$, set $a = (1+\gamma)/(\alpha-\delta)+\gamma$ and $b = 1/(\alpha+\delta)$, where α is as in (1.1.9). We want to prove that

$$\lim_{N\to\infty}\mathbb{P}(\tau^{\mathbf{2},\mathbf{1}}_N/N \geq a) = 0 \tag{4.3.12}$$

and

$$\lim_{N\to\infty}\mathbb{P}(\tau^{\mathbf{2},\mathbf{1}}_N/N \leq b) = 0. \tag{4.3.13}$$

To deal with (4.3.12) we write the probability of the event as follows

$$\begin{aligned}\mathbb{P}(\tau^{\mathbf{2},\mathbf{1}}_N/N \geq a) &= \sum_{k=1}^{\infty}\mathbb{P}(\tau^{\mathbf{2},\mathbf{1}}_N/N > a; G = k)\\ &= \sum_{k=1}^{\infty} 2\mathbb{P}(\tau^{\mathbf{2},\mathbf{1}}_N/N > a; G = k; \zeta^{\mathbf{2},\mathbf{1}}_{S_k}(1) = 2),\end{aligned} \tag{4.3.14}$$

where the second equality follows by the symmetry of the Harris construction. Thus, by Lemma 4.2 we have that the series above converges uniformly in N, which implies that

$$\lim_{N\to\infty}\mathbb{P}(\tau^{\mathbf{2},\mathbf{1}}_N/N \geq a) = \sum_{k=1}^{\infty}\lim_{N\to\infty} 2\mathbb{P}(\tau^{\mathbf{2},\mathbf{1}}_N/N > a; G = k; \zeta^{\mathbf{2},\mathbf{1}}_{S_k}(1) = 2).$$

Then, instead of (4.3.12) we prove that for all k

$$\lim_{N\to\infty}\mathbb{P}(\tau^{\mathbf{2},\mathbf{1}}_N/N > x; G = k; \zeta^{\mathbf{2},\mathbf{1}}_{S_k}(1) = 2) = 0. \tag{4.3.15}$$

In order to do that, we take $\epsilon > 0$ and for this ϵ we choose M such that

$$\mathbb{P}(M < S_k < \infty) \leq \epsilon$$

and $0 < M < \gamma N$ for N large enough. With these restrictions on M, we split the event in

(4.3.15) as follows

$$\begin{aligned}\mathbb{P}(\tau_N^{\mathbf{2,1}} \geq Na; G = k; \zeta_{S_k}^{\mathbf{2,1}}(1) = 2) &= \mathbb{P}(\tau_N^{\mathbf{2,1}} \geq Na; M < S_k < \infty; S_{k+1} = \infty; \zeta_{S_k}^{\mathbf{2,1}}(1) = 2) \\ &= \mathbb{P}(\tau_N^{\mathbf{2,1}} \geq Na; S_k < M; S_{k+1} = \infty; \zeta_{S_k}^{\mathbf{2,1}}(1) = 2).\end{aligned} \tag{4.3.16}$$

By our choice of M we have that the first term at the right member of (4.3.16) is less than $\epsilon/2$. Therefore, we focus on the second term. Our selection of M and a implies the following inequalities

$$\begin{aligned}&\mathbb{P}(\tau_N^{\mathbf{2,1}} \geq Na; \zeta_{S_k}^{\mathbf{2,1}}(1) = 2; S_k < M; S_{k+1} = +\infty) \\ &= \mathbb{P}((\tau_N^{\mathbf{2,1}} - S_k) > Na - S_k; \zeta_{S_k}^{\mathbf{2,1}}(1) = 2; S_k < M; S_{k+1} = +\infty) \\ &\leq \mathbb{P}((\tau_N^{\mathbf{2,1}} - S_k) > Na - M; \zeta_{S_k}^{\mathbf{2,1}}(1) = 2; S_k < M; S_{k+1} = +\infty) \\ &\leq \mathbb{P}(\tau_N^{\mathbf{2,1}} - S_k > N(1+\gamma)/(\alpha - \delta); \zeta_{S_k}^{\mathbf{2,1}}(1) = 2; S_{k+1} = +\infty),\end{aligned} \tag{4.3.17}$$

rewriting the last term above and using Remark 4.1 we obtain that

$$\begin{aligned}&\mathbb{P}(\tau_N^{\mathbf{2,1}} \geq Na; S_k < M; S_{k+1} = \infty; \zeta_{S_k}^{\mathbf{2,1}}(1) = 2) \\ &\qquad\leq \mathbb{P}(\tau_N^{\zeta_{S_k}^{\mathbf{2,1},N}}/N > (1+\gamma)/(\alpha-\delta); \zeta_{S_k}^{\mathbf{2,1}}(1) = 2; S_2^{\zeta_{S_k}^{\mathbf{2,1}}} = +\infty) \\ &\qquad\leq \mathbb{P}(\tilde{\tau}_N^{\zeta_{S_k}^{\mathbf{2,1},N}}/N > (1+\gamma)/(\alpha-\delta)).\end{aligned} \tag{4.3.18}$$

Remark 4.2 and Lemma 4.3 implies that the probability above converge to zero when N goes to infinity. Back to (4.3.16), we have that

$$\lim_{N\to\infty} \mathbb{P}(\tau_N^{\mathbf{2,1}}/N \geq x; G = k; \zeta_{S_k}^{\mathbf{2,1}}(1) = 2) \leq \lim_{N\to\infty} \mathbb{P}(\tilde{\tau}_N^{\zeta_{S_k}^{\mathbf{2,1},N}}/N > x) + \epsilon = \epsilon,$$

for an arbitrary ϵ and (4.3.12) is proved.

To conclude the theorem it remains to show (4.3.13). For $0 < \epsilon < \delta$, there exists T such that for all $t > T$

$$(\alpha - \epsilon) < r_t^{(-\infty,0]}/t < (\alpha + \epsilon) \text{ and } (\alpha - \epsilon) < -l_t^{[0,\infty)}/t < (\alpha + \epsilon), \tag{4.3.19}$$

almost surely. Now, take N large enough such that $bN > T$ and observe that

$$\begin{aligned}
&\{l_{bN}^{[0,-\infty)} > -(\alpha+\epsilon)bN; \xi^{\mathbb{1}}_{[-N+1,N]}(Nb) \cap [-N+1, -(\alpha+\epsilon)bN] \neq \emptyset\} \cap \\
&\{r_{bN}^{(-\infty,0]} < (\alpha+\epsilon)bN; \xi^{\mathbb{1}}_{[-N+1,N]}(Nb) \cap [(\alpha+\epsilon)bN, N] \neq \emptyset\} \\
&\subset \{\tau_N^{\mathbf{2},\mathbf{1}} > Nb\},
\end{aligned} \tag{4.3.20}$$

then

$$\begin{aligned}
\{\tau_N^{\mathbf{2},\mathbf{1}} \le Nb\} \subset \{&r_{bN}^{(-\infty,0]} \ge (\alpha+\epsilon)bN\} \cup \{l_{bN}^{[0,-\infty)} > -(\alpha+\epsilon)bN\} \\
&\cup \{\xi^{\mathbb{1}}_{[-N+1,N]}(Nb) \cap [-N+1, -(\alpha+\epsilon)N] = \emptyset; T^{\mathbb{1}}_{[-N+1,N]} > bN\} \\
&\cup \{\xi^{\mathbb{1}}_{[-N+1,N]}(Nb) \cap [(\alpha+\epsilon)bN, N] = \emptyset; T^{\mathbb{1}}_{[-N+1,N]} > bN\} \cup \{T^{\mathbb{1}}_{[-N+1,N]} \le bN\}.
\end{aligned} \tag{4.3.21}$$

The probabilities of the first two terms in the right member of (4.3.21) go to zero when N goes to infinity by (4.3.19). For the probability of the last term in (4.3.21), we use that $T_N/\mathbb{E}(T_N)$ converges in distribution to the exponential law with parameter 1. Also, it is known that $\mathbb{E}(T_N)$ is exponential on N, then we can conclude that

$$\limsup_{N\to\infty} \mathbb{P}(T^{\mathbb{1}}_{[-N+1,N]} < bN) = 0.$$

By the symmetry of Harris construction the probability of the fourth term in (4.3.21) is equal to the probability of the third. Therefore, it is enough to prove that the probability of the third term in (4.3.21) is small when N goes to infinity. To obtain this limit, first observe that by the path crossing property we have that

$$\min \xi^{\mathbb{1}}_{[-N+1,N]}(Nb) = \min \xi^{\mathbb{1}}_{[-N+1,\infty)}(Nb) \text{ in } \{T^{\mathbb{1}}_{[-N+1,N]} > Nb\}.$$

where $\xi^{\mathbb{1}}_{[-N+1,\infty)}$ is the contact process restricted to $[-N+1,\infty)$ with initial configuration full occupancy. By the translation invariance of the Harris graph, the argument above and the symmetry of the law of Harris graph, we have that

$$
\begin{aligned}
&\mathbb{P}(\min \xi^{\mathbb{1}}_{[-N+1,N]}(Nb) > -(\alpha+\epsilon)yN; T^{\mathbb{1}}_{[-N+1,N]} > Nb) \\
&= \mathbb{P}(\min \xi^{\mathbb{1}}_{[-N+1,\infty)}(Nb) > -(\alpha+\epsilon)bN; T^{\mathbb{1}}_{[-N+1,N]} > Nb) \\
&\leq \mathbb{P}(\min \xi^{\mathbb{1}}_{[1,\infty)}(Nb) > (1-(\alpha+\epsilon)b)N),
\end{aligned}
$$

where $1-(\alpha+\epsilon)b > 0$, by our choice of b and since ϵ is smaller than δ. By the fact that μ_+ is stochastically smaller than $\xi^{\mathbb{1}}_{[1,\infty)}$ we have the following inequality

$$
\begin{aligned}
\mathbb{P}(\min \xi^{\mathbb{1}}_{[1,\infty)}(Nb) > (1-(\alpha+\epsilon)b)N) &= \mathbb{P}(\min \chi^{\mathbb{N}}_{Nb} > (1-(\alpha+\epsilon)b)N) \\
&\leq \mu_+(\chi : \min \chi > (1-(\alpha+\epsilon)b)N).
\end{aligned}
$$

Thus, we have that

$$
\lim_{N\to\infty} \mathbb{P}(\max \xi^{\mathbb{1}}_{(-\infty,0]}(Nb) < -(1-(\alpha+\epsilon)b)N) = 0.
$$

By all the arguments above we can conclude the limit (4.3.13).

□

Future problems

There are many interesting questions for the contact process with two types of particles and priority that can be addressed in future works.

For example, we would like to study if there is a metastability behavior when the dynamic is restricted to $[0,1] \times [1,N]$, initial configuration $\mathbb{1}_{\{0\}\times[1,N]} + 2\mathbb{1}_{\{1\}\times[1,N]}$ and the particles of type 1 have priority in $\{0\} \times [1,N]$ and the particles of type 2 in $\{1\} \times [1,N]$. Even the case $R = 1$, we can not use the ideas presented in Chapter 3, because the boundary of interaction between the families of particles depends on N in this situation, different from the one in Chapter 3.

Also, we want to have more information about the set of invariant measures for the process. In Section 4.2.2, we only proved the existence of a non-trivial invariant measure different from μ, which is also an invariant measure for the process.

A challenging problem for us is the case when the dynamic is restricted to $[-N,N]^2$ with initial configuration $\mathbb{1}_{[-N,0]\times[-N,N]} + 2\mathbb{1}_{[1,N]\times[-N,N]}$ and the particles of type 1 have priority in $[-N,0] \times [-N,N]$ and the particles of type 2 in $[1,N] \times [-N,N]$. We want to understand if this model presents a metastability behavior. We need a different approach from the one used in Chapter 2 and Chapter 3, because in $\mathbb{Z}^2$ the dynamic of the contact process is more complicated.

Appendix

Details on Bezuidenhout-Grimmett renormalization

In this section we adapt the Bezuidenhout-Grimmett renormalization for a layer with fixed width $2L+1$. We prove the existence of a 1-dependent percolation system with closure below δ, Φ in $\{0,1\}^{\Lambda}$. The map Φ is defined using the contact process restricted to the layer $[-L,L]\times\mathbb{Z}$, with infection parameter $\lambda>\lambda_L$, where

$$\lambda_L=\inf\{\lambda:\mathbb{P}_\lambda(\xi_{[-L,L]\times\mathbb{Z}}(t)\neq\emptyset\,\forall t)>0\}.$$

This percolation system will be used in the next section to prove that outside an event with small probability, if the contact process in the layer survives for a time t, then at this time there is no empty segment (also called gap) of length $\sqrt{t}$.

In order to choose r, K and T such that Φ has closure close to 0 we present several results introduced in [3].

To simplify notation, during this section we refer to the contact process restricted to $B_L=[-L,L]\times\mathbb{Z}$, with initial configuration $A\subset[-L,L]\times\mathbb{Z}$, as $\xi^A_{B_L}(t)$. In the special case $A=[-L,L]\times\mathbb{Z}$ we write $\xi^{\mathbb{1}}_{B_L}(t)$. The fact that $\lambda>\lambda_L$ implies that for $\epsilon>0$ there exists r large enough such that

$$\mathbb{P}_\lambda(\xi^R_{B_L}(t)\neq\emptyset,\,\forall t)>1-\frac{1}{2}\epsilon^8, \tag{4.3.22}$$

where $R=[-L,L]\times[-r,r]$. In the first step we show that with large probability the rectangle R is connected to many points in the top and the faces of a large space time box. To enunciate this result, we need more notation:

For a space time box $B(K,S)=[-L,L]\times[-K,K]\times[0,S]$ we define $T_1(K,S)$ the number of points in $[-L,L]\times[0,K]\times\{S\}$ connected with $R\times\{0\}$ inside $B(K,S)$. $T_2(K,S)$ refers to the equivalent variable for the octant $[-L, L] \times [-K, 1] \times \{S\}$.

We define $F_1(K, S) = [-L, L] \times \{K\} \times [0, S]$ the right side of the box $B(K, S)$. Choose $h \in (0, (1+4\lambda)^{-1})$ and denote $N_1 = N_1(K, S)$ as the number of points in F_1 at distance h from each other, connected with $R \times \{0\}$ inside $B(K, S)$. We refer to the left side of the box as F_2 and N_2 is the corresponding variable. Also we define the total number of infected sites on the top of the box and at the sides as

$$T(K, S) = T_1(K, S) + T_2(K, S),$$

and

$$N_s(K, S) = N_1(K, S) + N_2(K, S),$$

respectively. Their sum is denoted by

$$N(K, S) = T(K, S) + N_s(K, S).$$

With $h \in (0, (1+4\lambda)^{-1})$, let α be the minimum of the probabilities of the following events

i) $(x, 0, 0)$ is connected to $R \times \{h\}$ inside $R \times [0, h]$ for every $x \in [-L, L]$,

ii) $(x, 0, 0)$ is connected to $R \times \{h\} \pm (0, r, 0)$ for every $x \in [-L, L]$.

Let M large enough to ensure that $(1-\alpha)^M \leq \epsilon$. We take N large enough such that in any subset of $\mathbb{Z}^2$ or $\mathbb{Z}$ having more than N elements, there are at least M points such that every pair of these points are at distance $3r + 1$ apart.

The proof of the following lemma is basically the same as that of Lemma (7) in [3]. The only difference is the fact that we are dealing with the contact process restricted to a layer and in the original paper their work for $\mathbb{Z}^2$ but all the arguments there are valid to our case.

Lemma 4.4. *For $\epsilon > 0$ there exist $M, N \in \mathbb{N}$, K and S such that*

$$\mathbb{P}(T_i(K, S) > 2N) > 1 - \epsilon, \tag{4.3.23}$$

and

$$\mathbb{P}(N_{F_i}(K, S) > 4MN) > 1 - \epsilon. \tag{4.3.24}$$

Proof. Observe that for all $t > 0$

$$\mathbb{P}(\xi^R_{B_L} \text{ dies} \,|\, |\xi^R_{B_L}(t)| < 2N) \geq (1+4\lambda)^{-2N}.$$

Therefore

$$\mathbb{P}(\xi^R_{B_L} \text{ survives} \,; |\xi^R_{B_L}(t)| < 2N \text{ for arbitrarily large times } t) = 0.$$

By our selection of R we have that

$$\mathbb{P}(\xi^R_{B_L} \text{ survives}; \exists\, T_1 \text{ such that } \forall t \geq T_1 |\xi^R_{B_L}(t)| \geq 2N) \geq 1 - \frac{1}{2}\epsilon^2,$$

then there exists T_1 such that

$$\mathbb{P}(\forall t \geq T_1,\ |\xi^R_{B_L}(t)| \geq 2N) \geq 1 - \frac{1}{2}\epsilon^2. \qquad (4.3.25)$$

From the equation above we have that for all $t \geq T_1$ there exists $K = K(t)$ such that

$$\mathbb{P}(T(K,t) \geq 2N) \geq 1 - \frac{2}{3}\epsilon^2. \qquad (4.3.26)$$

By FKG inequality we have that

$$\mathbb{P}(T_i(K,t) \leq N)^2 = \mathbb{P}(T_1(K,t) \leq N)\mathbb{P}(T_1(K,t) \leq N) \leq \mathbb{P}(T(K,t) \leq 2N) \leq \epsilon^2. \qquad (4.3.27)$$

Hence we concluded the first inequality in the Lemma for any $S \geq T_1$.

For the second equation, we define $s(K(t))$ as the infimum of the times such equation (4.3.26) is not satisfies. Because $\mathbb{P}(T(K,t) > 2N)$ is a continuous function of t, s is such that $\mathbb{P}(T(K,t) > 2N) = 1 - \epsilon^2$. We write B_k for the box of dimensions $k, s_k = s(k)$ with k going to infinity and $\mathbb{P}(T(k,s_k) > 2N) = 1 - \epsilon^2$. We write $N_k = T(k,s_k) + N_s(k,s_k)$. It is proved in the same way as in [3] that

$$\mathbb{P}(\forall k \geq k_0, N_k \geq 2N(2M+1)) > 1 - \epsilon^4, \qquad (4.3.28)$$

from (4.3.28) and our selection of s_k we have that

$$\begin{aligned}\epsilon^4 &\geq \mathbb{P}(T(k,s_k)+N_s(k,s_k)<2N(2M+1))\\ &\geq \mathbb{P}(T(k,s_k)<2N)\mathbb{P}(N_s(k,s_k)<4NM)\\ &=\epsilon^2\mathbb{P}(N_s(k,s_k)<4NM),\end{aligned}$$

which imply that

$$\mathbb{P}(N_s(k,s_k)>NM)\geq 1-\epsilon^2.$$

As in (4.3.27) we use FKG inequality to conclude that

$$\mathbb{P}(N_i(k,s_k)\leq 2NM)^2=\mathbb{P}(N_1(k,s_k)\leq 2NM)\mathbb{P}(N_2(k,s_k)\leq 2NM)\leq \mathbb{P}(N_s(k,s_k))\leq \epsilon^2. \tag{4.3.29}$$

To prove (4.3.28) the idea is first prove that for $\nu\in\mathbb{N}$ and $k\geq 1$

$$\mathbb{P}(\xi^R_{B_L} \text{ dies out } |N_k\leq \nu)\geq \left(\frac{1-4\lambda h}{1+4\lambda}\right)^{\nu}, \tag{4.3.30}$$

ones we have this inequality (4.3.28) runs as (4.3.25).

To obtain (4.3.30) we first observe that for every x on the top of B_k we prevent the spread of the infection through the line $x\times[s_k,\infty)$ if it happens a mark of death before any mark of infection, this has probability $1/(1+4\lambda)\geq(1-4\lambda h)/(1+4\lambda)$. Now we estimate the probability of the event there is non spread of infection through the sides of B_k. For z in $[-L,L]\times\{-k\}\cup[-L,L]\times\{k\}$ let us organize, by the increasing order of the second coordinates, the set of points in $\{z\}\times[0,s_k]$ connected with $R\times\{0\}$ inside the interior of B_k and denoted them by $(z,p_1),\ldots,(z,p_n)$. Divided the points (z,p_i) into groups $\{(z,p_{i_l}),\ldots,(z,p_{j_l})\}$ satisfying (i) and (ii) below

i) $p_{j_l}-p_{i_l}<h$,

ii) $p_{i_{l+1}}-p_{i_l}>h$.

Let $m(z)$ the number of groups that we can construct with these characteristics. Also denote $i_l(\ j_l-1)$ the small(higher) index such that $(z,p_{i_l})(\ (z,p_{j_l-1}))$ belong to the l-th group. The probability that there is non infection trough the segment $\{x\}\times(p_{i-1},p_i)$ to outside B_k is given by

$$\frac{1}{1+4\lambda} + \frac{4\lambda}{1+4\lambda} e^{-(1+4\lambda)y_i},$$

where $y_i = p_i - p_{i-1}$. Then the probability that there is non infection leaving B_k trough the line $\{x\} \times [0, s_k]$ is given by

$$\prod_{i=1}^{n} \frac{1}{1+4\lambda} + \frac{4\lambda}{1+4\lambda} e^{-(1+4\lambda)y_i} \geq \prod_{\tau=1}^{m} \frac{1}{1+4\lambda} \prod_{l=1}^{j_\tau} \left(\frac{1}{1+4\lambda} + \frac{4\lambda}{1+4\lambda} e^{-(1+4\lambda)y_l} \right),$$

in order to estimate this quantity observe that

$$\begin{aligned}
(1+4\lambda)^{-1}(1+4\lambda e^{-(1+4\lambda)y_l}) &= (1+4\lambda)^{-1} + (1+4\lambda)^{-1} 4\lambda e^{-y_l} e^{-4\lambda y_l} \\
&\geq (1+4\lambda)^{-1} e^{-4\lambda y_l} + \frac{4\lambda}{(1+4\lambda)} e^{-h} e^{-4\lambda y_l} \\
&\geq e^{-4\lambda y_l} \left(\frac{1}{(1+4\lambda)} + \frac{4\lambda}{(1+4\lambda)} (1-h) \right) \\
&\geq e^{-4\lambda y_l} \left(\frac{1}{(1+4\lambda)} + \frac{(4\lambda)^2}{(1+4\lambda)} \right) \\
&\geq e^{-4\lambda y_l},
\end{aligned}$$

where the second inequality use the fact that $e^{-x} \geq (1-x)$, the third is because $h \leq (1+4\lambda)^{-1}$ and the last inequality use that $\lambda > \frac{1}{4}$. Therefore, we obtain that

$$\begin{aligned}
\prod_{\tau=1}^{m} \prod_{l=1}^{j_\tau} \frac{1}{1+4\lambda} \left(\frac{1}{1+4\lambda} + \frac{4\lambda}{1+4\lambda} e^{-(1+4\lambda)y_l} \right) &\geq \left(\frac{1}{1+4\lambda} \right)^m \prod_{\tau=1}^{m} \prod_{l=i_\tau}^{j_\tau - 1} e^{-4\lambda y_l} \\
&\geq \left(\frac{1}{1+4\lambda} \right)^m \prod_{\tau=1}^{m} (1 - 4\lambda \Sigma_{l=i_\tau}^{j_\tau - 1} y_l) \\
&\geq \left(\frac{1-4\lambda h}{1+4\lambda} \right)^m,
\end{aligned}$$

the second inequality above use again that $e^{-x} \geq (1-x)$. Running over all x in the sides we have that the probability that the infection do not leave B_k trough the sides of the box is at least

$$\prod_{z\in[-L,L]\times\{-k\}\cup[-L,L]\times\{k\}} \left(\frac{1-4\lambda h}{1+4\lambda}\right)^{m(z)} \geq \left(\frac{1-4\lambda h}{1+4\lambda}\right)^{N_s(s_k,k)}.$$

Now it is easy to conclude (4.3.30), finishing the lemma. □

In the second step we construct a "seed" (a fully occupied translate of the rectangle R) in the side of the box $B(K,S)$. From this seed we iterated Lemma 4.4. The following Lemma is the equivalent to Lemma (18) in [3]. We rewrite the proof of the original lemma because we have a restriction in the first coordinate.

Lemma 4.5. *Suppose $\mathbb{P}(\xi^0_{B_L}$ survives$) > 0$ and $\epsilon > 0$. There exists r, K and T such that the following holds: with $\mathbb{P}_\lambda$-probability greater than $1-\epsilon$, there exists a translate Δ of the rectangle R, as in (4.3.22) such that*

- *(i)* $\Delta \subset [-L,L]\times[K,2K]$,
- *(ii) There exists $t\in[T,2T]$ such that $R\times\{0\}$ is connected inside $[-L,L]\times[-K,3K]\times[0,2T]$ to every point in $\Delta\times\{t\}$.*

Proof. With r as in (4.3.22), $h\in(0,(1+4\lambda)^{-1})$, M and N defined as before. Also we choose K and S as in Lemma 4.4 and we set $T=S+h$. By Lemma 4.4 we have that, with probability larger than $1-\epsilon$, there exists NM points connected with R in the side F_3 of the box $B(K,S)$, and this points are at distance h. There exists at least one line $(x,K)\times[0,S]$ such that it has at least M points at distance h from each other. Then with probability larger than $1-\epsilon$ we have that there exists one time s such that (x,K,s) is connected to $(R+(0,r+K))\times\{s+h\}$. Define τ as the smaller time such that $R\times\{0\}$ is connected to a translate $R'=[-L,L]\times[K,K+2r]$. By our discussion above we have that the probability that $\tau\in[0,S+h]$ is larger than $(1-\epsilon)^2$.

Using the Strong Markov Property and Lemma 4.4, we have that $R'\times\{\tau\}$ is connected with N points in $[-L,L]\times[K+r,2K+r]\times\{\tau+S\}$.

From those N points we select M points such that each pair of points is at distance at least $3r+1$. We denote this subset of points as A. For every $z=(x,y)\in A$ with the second coordinate less than $2K$ we associate the cylinder $R_y\times[\tau+S,\tau+S+h]$. For every $z=(x,y)\in A$ with the second coordinate greater than $2K$ we associate the cylinder

$(R_y - (0,r)) \times [\tau + S, \tau + S + h]$. These cylinders are disjoint, and by our selection of M we have that with probability larger than $(1-\epsilon)$ at least one $z = (x,y) \in A$ satisfies $(z, \tau + S) \to z' \times \{\tau + S + h\}$ inside $\tilde{R} \times [\tau + S, \tau + S + h]$ for every $z' \in \tilde{R}$, where $\tilde{R}$ is R_y or $R_y - (0,r)$ depending on the second coordinate of z. Therefore, we get a translation of R in $[-L,L] \times [K,2K] \times [S+h, 2(S+h)]$, connected inside $[-L,L] \times [-K,3K] \times [0,2T]$.

With probability greater than $(1-\epsilon)^4$, the construction above give a translation $\Delta \times \{s\}$ of $R \times \{0\}$ such that $\Delta \times \{s\} \subset [-L,L] \times [K,2K] \times [T,2T]$. □

The final step is to repeat Lemma 4.5 many times in order to enlarge the dimension of the renormalized box with the aim that $\Psi^{\mathrm{bg}}(m,n)$ will be a 1-dependent system. In the following lemma we use the notation $R_y = [-L,L] \times [-r,r] + (0,y)$.

Lemma 4.6. *For any $\epsilon > 0$, $j \in \mathbb{N}$, $y \in [-2K,2K]$ and $t \in [0,2T]$ with $\mathbb{P}_\lambda$-probability larger than $(1-\epsilon)^{2j}$, there exists a translation of $R_y \times \{t\}$, Π, such that*

(i) $\Pi \subset [-L,L] \times [-2K + jK, 2K + jK] \times [j2T, (j+1)2T)]$,

(ii) *$R_y \times \{t\}$ is connected to all points in Π inside the region*

$$\bigcup_{0 \le i \le j-1} [-L,L] \times [-3K + iK, 4K + iK] \times [0, 2T + i2T]. \qquad (4.3.31)$$

Proof. The proof of this lemma is basically the proof of Lemma (19) in [3].

We first prove the Lemma for $j = 1$. To obtain the result we use Lemma 4.5 and Strong Markov property.

Case 1 $y \in [-2K, K]$ and $t \in [T, 2T]$. By Lemma 4.4 we have that with probability larger than $(1-\epsilon)$ there exists a translation of $R_y \times \{t\}$ $[-L,L] \times [K+y, 2K+y] \times s$ with $s \in [T+t, 2T+t]$, such that every point in this translation is connected with $R_y \times \{t\}$ inside $[-L,L] \times [-K+y, 3K+y] \times [t, 2T+t]$. This way we have a translation Π in $[-L,L] \times [-K,3K] \times [2T,4T]$ with all points connected with $R_y \times \{t\}$ inside $[-L,L] \times [-3K,4K] \times [0,4T]$, with probability larger than $(1-\epsilon)$.

Case 2 $y \in [K, 2K]$ and $t \in [T, 2T]$. By Lemma 4.4 we have that with probability larger than $(1-\epsilon)$ there exists a translation of $R_y \times \{t\}$ in $[-L,L] \times [-2K+y, -K+y] \times s$ with $s \in [T+t, 2T+t]$, such that every point in this translation is connected with

$R_y \times \{t\}$ inside $[-L, L] \times [-3K + y, K + y] \times [t, 2T + t]$. This way we have a translation Π in $[-L, L] \times [-K, K] \times [2T, 4T]$ with all points connected with $R_y \times \{t\}$ inside $[-L, L] \times [-2K, 3K] \times [0, 4T]$, with probability larger than $(1 - \epsilon)$.

Case 3 $y \in [-2K, 0]$ and $t \in [0, T]$. By Lemma 4.4 we have that with probability larger than $(1 - \epsilon)$ there exists a translation of $R_y \times \{t\}$ in $[-L, L] \times [K + y, 2K + y] \times s$ with $s \in [t, T + t]$, such that every point in this translation is connected with $R_y \times \{t\}$ inside $[-L, L] \times [-K + y, 3K + y] \times [t, T + t]$. This way we have a translation Π in $[-L, L] \times [-K, 2K] \times [T, 2T]$ with all points connected with $R_y \times \{t\}$ inside $[-L, L] \times [-3K, 3K] \times [0, 2T]$. Since the center of the translation Π is in $[-L, L] \times [-K, 2K] \times [T, 2T]$ we can apply Case 1 or Case 2 to find a translate of Π in $[-L, L] \times [-K, 3K] \times [2T, 4T]$ connected inside $[-L, L] \times [-3K, 4K] \times [0, 4T]$. The composite of the two steps is successful with P_λ-probability greater than $(1 - \epsilon)^2$.

The other cases are very similar to Case 3.

For $j \geq 2$ we iterated the construction above and use the Strong Markov property. □

Taking $j = 11$ in Lemma 4.5 we obtain the following proposition.

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